What do fund flows reveal about asset pricing models and investor sophistication?

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What do fund flows reveal about asset pricing models and investor sophistication?

Recent literature uses the relative strength of the relation between fund flows and alphas with respect to various multifactor models to draw inferences about the best asset pricing model and about investor sophistication. This paper analytically shows that such inferences are tenable only under certain assumptions and we test their empirical validity. Our results indicate that any inference about the true asset pricing model based on alpha-flow relations is empirically untenable. The literature uses a multifactor model that includes all factors as the benchmark to assess investor sophistication. We show that the appropriate benchmark excludes some factors when their betas are estimated from the data, but even with this benchmark the rejection of investor sophistication in the literature is empirically tenable.

An extensive literature documents that net fund flows into mutual funds are driven by funds' past performance. For example, Patel, Zeckhauser, and Hendricks (1994) document that equity mutual funds with bigger returns attract more cash inflows and they offer various behavioral explanations for this phenomenon. Other papers that document a positive relation between fund flows and past performance include Ippolito (1992), Chevalier and Ellison (1997), and Sirri and Tufano (1998).

Some papers in the early literature also examine whether abnormal performance (or alphas) measured with respect to some benchmarks better predict fund flows than others. For example, Gruber (1996) compares the mutual fund flow-performance relation for alphas measured with respect to one- and four-factor models, while Del Guercio and Tkac (2002) compares sensitivity of flows to raw returns vis-à-vis alphas from market model in mutual funds and pension funds. Fung et. al. (2008) makes similar comparisons with a different set of factor models for a sample of hedge funds.

While comparison of flow-alpha relations across models was not the primary focus of earlier papers, recent papers in this area have shown a renewed interest in such comparisons using a broader range of asset pricing and factor models. The primary driving force for this resurgence is the argument that these comparisons can potentially help us answer important economic questions that extend beyond a descriptive analysis of mutual fund flows. For example, Barber, Huang and Odean (2016) (hereafter "BHO") compare the relation between fund flows and alphas measured with respect to various models to evaluate mutual fund investors' sophistication. They argue that sophisticated investors should use all common factors to compute alphas and evaluate fund performance regardless of the underlying true asset pricing model. BHO find that fund flows are more highly correlated with market model alphas than with other alphas. Because investors do not seem to be using alphas with respect to a model that includes all common factors, BHO conclude that investors in aggregate are not sophisticated in how they use past returns to assess fund performance.

Berk and van Binsbergen (2016) (hereafter "BvB") argue that such comparisons serve as a new and fundamentally different test of asset pricing models and that the results can determine which asset pricing model is the closest to the true asset pricing model in the economy. Because of the asset pricing model implications, they include several versions of equilibrium consumption-

CAPM as well in their comparisons. Agarwal, Green and Ren (2017) and Blocher and Molyboga (2017) carry out similar tests with samples of hedge funds.

BvB find that fund flows are most highly correlated with alphas computed with a market model in their tests as well. They conclude that therefore the CAPM "is still the best method to use to compute the cost of capital of an investment opportunity." Berk and van Binsbergen (2017) also prescribe that practitioners should use the CAPM to make capital budgeting decisions based on this evidence. The true asset pricing model has been a holy grail of the finance literature and BvB's conclusions potentially have broad implications that go well beyond just the mutual fund literature.

The far reaching inferences drawn in the recent literature based on comparisons of flow-alpha relations stand in contrast with the much more limited inferences drawn in the early literature. A natural question that arises is, under what assumptions can one draw reliable inferences about asset pricing models or about investor sophistication based on these results? Are the inferences about asset pricing models and investor sophistication in the recent literature empirically tenable?

We address these questions in this paper. We analytically show that one can draw reliable inferences about the true asset pricing model based on flow-alpha relations only if certain critically important assumptions are valid, and their validity can only be empirically determined. For example, it is possible that in some situations CAPM may not be true but investors may still optimally use the market model to estimate alphas. Also, in some other situations, it is possible that CAPM may be true but investors may optimally use a multifactor model to estimate alphas. There are also situations where investors may optimally use the market model to estimate alpha when CAPM is true, which would justify inferences about asset pricing model. Therefore, one cannot identify the true asset pricing model solely based on flow-alpha comparison without further tests to determine which of these multiple possibilities are true in the data.

We find similar issues with drawing inferences about investor sophistication as well. Sophisticated investors would use the model that yields the most precise alpha estimates. We show that the optimal model depends on the following factors: the underlying true asset pricing model, the incremental explanatory power of each factor in a multifactor model, the dispersion of factor betas across funds and the potential error in estimating factor betas. Our results indicate that this optimal model need not be the true asset pricing model, nor does it need to use all common factors to estimate betas. Therefore, the optimal model can only be empirically identified and we need the identity of this model to draw reliable inferences about investor sophistication based on flow-alpha relations.

We empirically assess whether inferences about asset pricing models and investor sophistication based on flow-alpha relations are tenable. Our tests estimate the relevant parameters from the data and run simulation experiments under various "true" asset pricing models. These tests enable us to determine the multifactor model that provides the most precise estimator of alphas in the data and assess the tenability of the inferences about asset pricing and investor sophistication in the literature.

1. Fund flows and alphas: Foundation for empirical tests and inferences

This section presents a model that forms the basis for our analysis of the implications of flow-alpha relations for asset pricing models and tests of investor sophistication. Broadly, we use the model to answer the following questions:

(a) How do investors optimally update their priors about the skills of fund managers when they observe fund returns each period?

(b) How are equilibrium fund flows related to the information investors use to update their priors?

(c) What are the implications of the answers to the above questions for interpreting the results of an alpha-fund flow horse race with alphas computed using different multifactor models?

We answer these questions using the Berk and Green (2004) model augmented with a multifactor return generating process and an equilibrium asset pricing model that we describe in the next subsection.

1.1 Return generating process and asset pricing model

The following K-factor model is the true asset pricing model:

$$E[r_i] = \sum_{k=1}^{K} \beta_{k,i}\gamma_k, \tag{1}$$

where r_i is the return in excess of the risk-free rate, or excess returns, $E[r_i]$ is the expected excess return on asset *i*, $\beta_{k,i}$ is the beta of asset *i* with respect to factor *k*, and γ_k is the premium for a unit of factor risk. For the CAPM, $K = 1$ and for Fama-French three-factor model, which we refer to as FF3, $K = 3$.

Asset returns follow the *J*-factor model below:[1]

$$r_{i,t} = E[r_i] + \sum_{k=1}^{J} \beta_{k,i} f_{k,t} + \xi_{i,t}, \tag{2}$$

where $f_{k,t}$ is the realization of the common factor *k*, and $\xi_{i,t}$ asset specific return at time *t*. Factor realization $f_{k,t}$ is the innovation or the unexpected component of factor *k*. For instance, let $F_{k,t}$ be the total factor realization of the k^{th} factor, then $f_{k,t} = F_{k,t} - E[F_{k,t}]$ and $E[f_{k,t}] = 0$. We define a "no-beta risk premium" (NBRP) model where $E[r_i] = E[r_m]$, and we identify this model with $K = 0$.

In general the J factors in the multifactor model (3) include the K priced factors from the asset pricing model as well as additional unpriced factors that describe realized returns but are excluded from the asset pricing model. For example, the J factors could include industry factors that are unpriced perhaps because they are not correlated with future investment opportunity set or to consumption. Therefore, in general $J \geq K$. Factor returns and asset specific returns are all normally distributed.

1.2 **The Model**

This subsection presents a rational expectations model that identifies the alphas that investors use to make their mutual fund investment decision. The following are our assumptions:

(a) **Rational Economy:** All agents in the rational expectations economy are symmetrically informed.

(b) **Mutual funds and skill:** There are *N* mutual funds in the economy and $N \longrightarrow \infty$. Manager of fund p is endowed with stock selection skills that allow them to generate gross returns

[1]Eq. 3 imposes the condition that the intercept of the return generating process for each asset equals its expected return.

of ϕ_p in excess of the *K*-factor asset pricing benchmark. Investors know the true asset pricing model. Fund manager skill $\phi_p \sim N(\phi_0, \nu)$, where ϕ_0 is average skill and ν is the precision of the distribution of skill at time 0. ϕ_0 and ν are common knowledge.

(c) **Costs of active management:** Funds incur certain costs for active management, which is a function of total assets under management, denoted as q, and $c(q)$ is this cost per unit of assets. The cost $c(q)$ includes both fund fees and the costs of administration and trading including administrative costs, brokerage costs and price impact. There are diseconomies of scale and hence $c(q)$ is an increasing function of q. The only further assumption that we impose on $c(q)$ is that $\lim_{q \to \infty} c(q) = \infty$, which ensures that the size of a mutual fund for any skill level is finite.

(d) **Gross and net returns:** Let $R_{p,t}$ and $r_{p,t}$ be fund *p*'s gross and net returns at time *t*, respectively. $R_{p,t} = r_{p,t} + c(q_{p.t-1})$. Funds' net returns are observable, both to investors in the model economy and to econometricians. Investors can also compute $R_{p,t}$ since they know q and $c(q)$ but econometricians observe only $r_{p,t}$.

(e) **Expected return and return generating process:** Eqs. (1) and (2) specify the expected returns and the return generating process in this economy, which are both common knowledge. The net return at time *t* is:[2]

$$r_{p,t} = \phi_p + \underbrace{\sum_{k=1}^{K} \beta_{k,p} \gamma_k}_{\text{Expected return Eq. (1)}} + \underbrace{\sum_{k=1}^{J} \beta_{k,p} f_{k,t} + \xi_{p,t}}_{\text{Unexpected return Eq. (2)}} - c(q_{p,t-1}), \tag{3}$$

From Eq. (3), expected abnormal return based on the true asset pricing model is:

$$\mathrm{E_t}(r_{p,t+1}) - \sum_{k=1}^{K} \beta_{k,p} \gamma_k = \mathrm{E_t}(\phi_p) - c(q_{p,t}), \tag{4}$$

where E_t denotes expectation as of time *t*.

(f) **Competitive Market:** As in Berk and Green (2004), the mutual fund market is perfectly competitive. Therefore, expected alpha net of fees and costs for investing in any mutual fund equals zero in equilibrium:

[2] Funds' gross returns follow the return generating process (2), plus ϕ_p^K. Investors earn net returns in (3) after all costs.

$$\mathrm{E_t}(\phi_p) - c(q_{p,t}) = 0. \tag{5}$$

Assumptions (a) through (d) and (f) are the same as in Berk and Green (2004). We add assumption (e) about expected asset returns and return generating factor model to allow investors the option of computing alphas using different models and then determine which of these alphas will win the alpha-flow horse race.

In our model, investors observe mutual fund returns and the realized returns on the *J* factors each period. For now, we assume $\beta_{k,p}$ is common knowledge. Investors compute alphas, $\hat{\alpha}_{p,\eta,t}$, relative to an η-factor model, as:

$$\hat{\alpha}_{p,\eta,t} = r_{p,t} - \sum_{k=1}^{\eta} \beta_{k,p} F_{k,t}\,, \tag{6}$$

where $F_{k,t}$ is realized factor returns.

Investors could possibly use any η-factor model to compute alphas and update their priors about fund manager skills. Which particular model would they use? For analytic convenience, we further assume that the average factor betas of funds equal betas for the market portfolio. For the market, $\beta_{k,market} = 1$ for $k =$ market and $\beta_{k,market} = 0$ for $k \neq$ market. With this assumption, the average mutual fund alpha estimated with any η equals zero for any factor as we show in the following proposition.[3]

Lemma 1: Under the assumptions stated above the cross-sectional average of alphas estimated using any η-factor model equals zero for all factor realizations, i.e.

$$\tfrac{1}{N}\sum_{p=1}^{N} \left(\hat{\alpha}_{p,\eta,t} \,\middle|\, F_{k,t}, k = 1, \dots, K\right) = 0 \;\forall\, \eta. \tag{7}$$

Proof: Substitute $r_{p,t}$ from Eq. (3) to Eq. (6) and use the competitive market condition (5) to get (7).

[3] If the average factor betas for funds are different from market portfolio betas, we could subtract the average of alpha for all funds from $\hat{\alpha}_{p,\eta,t}$ and the average of this difference equals zero for any factor realization. Since investment decisions are made based on the relative values of alpha both approaches would yield the same result.

The following proposition presents the distribution of investors' posterior each period conditional on using a particular η-factor model to compute alphas.

Proposition 1: Suppose investors use an η-factor model to compute alphas. Let $\phi_{p,\eta,t}$ be the mean of investors' time *t* posterior of fund *k*'s skill conditional on the realization of $X_{p,\eta,1}, X_{p,\eta,2}$, ... $X_{p,\eta,t}$, where $X_{p,\eta,t} = \hat{\alpha}_{p,\eta,t} + c(q_{p,t-1})$, and let $\bar{X}_{p,\eta,t}$ be the mean of these realizations. Investors' posterior of ϕ_p is normally distributed with mean $\phi_{p,\eta,t}$, where:

$$\phi_{p,\eta,t} = \frac{\nu\,\phi_0 + t\vartheta_{\hat{\alpha},\eta}\,\bar{X}_{p,\eta,t}}{\nu + t\vartheta_{\hat{\alpha},\eta}}, \tag{8}$$

and precision $\nu + t\vartheta_{\hat{\alpha},\eta}$, where $\vartheta_{\hat{\alpha},\eta} = \frac{1}{\sigma^2_{\hat{\alpha},\eta}}$. Note that the precisions of $X_{p,\eta,t}$ and $\hat{\alpha}_{p,\eta,t}$ are equal conditional on information available at time *t-1* since $c(q_{p,t-1})$ is known at that time.

Proof: See Theorem 1 in DeGroot (1970, p. 167).

Proposition 1 is a well-known theorem in Bayesian analysis. Investors know that the average skill of fund managers is ϕ_0 at time 0. After observing net returns for *t* periods, investors compute alphas for each period. As Lemma 1 shows, the mean of $\hat{\alpha}_{p,\eta,t}$ is zero each period. Investors also know the cost $c(q_{t-1})$ each period and therefore they can compute $X_{p,\eta,t}$. Because of the perfect competition assumption (5) the expected value of $c(q)$ in $\bar{X}$ across all funds equals ϕ_0. But mean of both investors prior and posterior equal ϕ_0 as long as funds do not enter or exit the sample.

Proposition 1 shows that the precision of the posterior distribution of $\phi^K_{p,\eta,t}$ increases monotonically with an increase in the precision of $\hat{\alpha}_{p,\eta,t}$. Because rational investors prefer a more precise estimator to a less precise estimator, investors would use the multi factor model that yields the most precise estimate of alphas.

Corollary: Rational investors would use the η-factor model with the smallest variance (or largest precision) to revise their priors about fund skills.

We denote the number of factors in the factor this optimal model as η^*.

1.3 Alphas and fund flows

The literature typically runs the following regression between alphas and fund flows to draw inferences about asset pricing model or investor sophistication:

$$\Gamma_{p,t} = a_\eta + b_\eta \times \hat{\alpha}_{p,\eta,t} + \omega_{p,\eta,t}, \tag{9}$$

where $\hat{\alpha}_{p,\eta,t}$ is computed under various η-factor models in Eq. (6) and $\Gamma_{p,t}$ is the net flow of funds into mutual fund p, defined as:

$$\Gamma_{p,t} = \frac{q_{p,t} - q_{p,t-1}\left(1 + r_{p,t}\right)}{q_{p,t-1}} = \frac{q_{p,t} - q_{p,t-1}}{q_{p,t-1}} - r_{p,t}. \tag{10}$$

BvB, BHO and others in related literature run a horse race with the slope coefficient b_η from (9) and draw inferences about the true asset pricing models and investors sophistication based on the winner.[4] This subsection derives fund flows in the model as a function of the alpha investors use to update their priors and examines the tenability of such inferences.

As we showed in the last subsection, investors update their priors each period using the most precise alpha estimator $\hat{\alpha}_{p,\eta^*,t}$ and decide on which funds they would invest in or divest from each period. The competitive market condition in Eq. (5) implies that in a competitive equilibrium the flow results in $q_{p,t}$ such that $c(q_t)$ equals $\phi_{p,\eta^*,t}$. To determine the fund flow in this competitive equilibrium, we use the following recursive equation for $c(q_t)$ that follows from Eq. (8) in Proposition 1:[5]

$$c(q_t) = c(q_{t-1}) + \frac{\vartheta_{\hat{\alpha},\eta^*}}{\nu + t\vartheta_{\hat{\alpha},\eta^*}} \times \hat{\alpha}_{p,\eta^*,t}. \tag{11}$$

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[4] BvB use a regression that uses only the signs of $\Gamma_{p,t}$ and $\hat{\alpha}_{p,\eta,t}$. We show in a later section that BvB regression yields identical inference as the linear regression (9).

[5] Berk and Green (2004) derive this equation based on DeGroot's (1970) theorem. Appendix 3 presents a derivation for easy reference.

To determine a functional relation between flow and alpha, we assume that the cost function is given by:

$$c(q) = \delta \times q, \tag{12}$$

where δ is a constant that is common knowledge. Using this cost function, from (10) and (11) we get:

$$\Gamma_{p,t} = \frac{1}{\delta \times q_{t-1}} \frac{\vartheta_{\hat{\alpha},\eta^*}}{\nu + t\vartheta_{\hat{\alpha},\eta^*}} \times \hat{\alpha}_{p,\eta^*,t} - r_{p,t}. \tag{13}$$

Eq. (13) shows that flows are directly related to $\hat{\alpha}_{p,\eta^*,t}$, the most precise estimate. As we show later, the winner of the horse race also depends on the sign of the covariance between $\Gamma_{p,t}$ and $\hat{\alpha}_{p,\eta^*,t}$. Intuitively, we expect that Covariance$\left(\Gamma_{p,t}, \hat{\alpha}_{p,\eta^*,t}\right) > 0$ but we need to show that it is indeed positive in our model. The first term on the right-hand side of Eq. (13) shows that $\Gamma_{p,t}$ increases with $\hat{\alpha}_{p,\eta^*,t}$ because its coefficient is positive. However, because the second term equals $-\left(\hat{\alpha}_{p,\eta^*,t} + \sum_{k=1}^{\eta^*} \beta_{k,p} F_{k,t}\right)$ from Eq. (6), there is a negative relation between this term and $\Gamma_{p,t}$. The reason is that $\Gamma_{p,t}$ is the flow net of changes in assets under management due to raw returns and any change due to return has a negative impact on $\Gamma_{p,t}$. So it is mathematically possible that the net effect would be negative.

As a first step, we examine the relation between $\Gamma_{p,t}$ and $\hat{\alpha}_{p,\eta^*,t}$. Differentiating (13), we get:[6]

$$\frac{d\,\Gamma_{p,t}}{d\hat{\alpha}_{p,\eta^*,t}} = \frac{1}{\delta q_{t-1}} \times \frac{\vartheta_{\hat{\alpha},\eta^*}}{\nu + t\vartheta_{\hat{\alpha},\eta^*}} - 1. \tag{14}$$

Eq. (14) indicates that the slope is negative if:

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[6] We use the result that $\frac{dr_{p,t}}{d\hat{\alpha}_{p,\eta^*,t}} = 1$ from (6).

$$\frac{\vartheta_{\hat{\alpha},\eta^*}}{\nu + t\vartheta_{\hat{\alpha},\eta^*}} < \delta q_{t-1}. \tag{15}$$

Eq. (11) shows that fund size would increase when $\hat{\alpha}_{p,\eta^*,t} > 0$ because positive alpha leads to upward revision of priors. When $\hat{\alpha}_{p,\eta^*,t} = 0$, the fund size does not increase in equilibrium, which implies that $\Gamma_{p,t} = 0$ for $\hat{\alpha}_{p,\eta^*,t} = 0$. So if $\frac{d\Gamma_{p,t}}{d\hat{\alpha}_{p,\eta^*,t}} < 0$, there would be negative flow in equilibrium when $\hat{\alpha}_{p,\eta^*,t} > 0$.

The right hand side of inequality (15) is the cost of active management per unit of fund. This condition indicates that if assets under management is sufficiently large, then because of the incremental cost of active management, investors would actually withdraw funds (or the mutual fund would voluntarily return funds) even if investors positively update their priors about fund manager skills. Therefore, funds in this economy cannot grow beyond the critical size defined by Eq. (15). Formally, from (15) the maximum size Q^{max} for a fund in this economy is:

$$Q^{max} = \frac{1}{\delta}\frac{\vartheta_{\hat{\alpha},\eta^*}}{\nu + t\vartheta_{\hat{\alpha},\eta^*}}, \tag{16}$$

because beyond this point any increase in fund size, even if it is passively due to fund size, will result in outflow of funds.[7] Since the maximum fund size is given by (16), in this economy,

$$\frac{d\Gamma_{p,t}}{d\hat{\alpha}_{p,\eta^*,t}} > 0 \text{ for } q_t \leq Q^{max}. \tag{17}$$

Proposition 2 below formally states the results from our model that we will use to examine what we can learn from the horse race regressions.

Proposition 2: The net flow of funds $\Gamma_{p,t}$ in any period is only a function of alpha from the most precise estimator and not from other estimators. Also, in equilibrium:

$$Cov(\hat{\alpha}_{p,\eta^*,t}, \Gamma_{p,t}) > 0 \tag{18}$$

[7] Historically, a few mutual funds have been closed to new investors evidently because they hit against the maximum assets under management condition and were not willing to accept additional assets. Several successful hedge funds are also closed to new investors and a well-known example is Renaissance Technologies which returned all funds from its outside investors by 2005 (https://en.wikipedia.org/wiki/Renaissance_Technologies#Monemetrics).

Proof: Eq. (13) proves the first part of the proposition. See Appendix 5 for proof of the second part.

1.4 Alpha-fund flows horse race

The literature fits regression (9) using estimates of $\hat{\alpha}_{p,\eta,t}$ from competing η-factor models and runs a horse race with the slope coefficient $\hat{b}_{\eta}$. What can we learn about the true asset pricing model or about investor sophistication based on this horse race?

The slope coefficient estimate is:

$$plim\ b_{\eta} = \frac{Cov(\Gamma_p, \hat{\alpha}_{p,\eta,t})}{\sigma^2_{\hat{\alpha}_{p,\eta,t}}} \tag{19}$$

where $\sigma^2_{\hat{\alpha}_{p,\eta,t}}$ is cross-sectional variance of $\hat{\alpha}_{p,\eta,t}$. As Proposition 2 shows Γ_p is a function of $\hat{\alpha}_{p,\eta^*,t}$ and therefore any covariance of Γ_p with other alphas is through their relations with $\hat{\alpha}_{p,\eta^*,t}$. Since the variance of alpha estimates from the less-than-optimal estimators are bigger than that from the optimal estimator, we can express other estimators as:

$$\hat{\alpha}_{p,\eta,t} = \hat{\alpha}_{p,\eta^*,t} + \zeta_{p,\eta,t}, \tag{20}$$

where $Var(\zeta_{p,\eta,t}) > 0$ for $\eta \neq \eta^*$, and $Cov(\zeta_{p,\eta,t}, \hat{\alpha}_{p,\eta^*,t}) = 0$. Because Γ_p is correlated with $\hat{\alpha}_{p,\eta^*,t}$ through Eq. (13) and not with noise in the less efficient estimates,

$$Cov(\Gamma_p, \hat{\alpha}_{p,\eta,t}) = Cov(\Gamma_p, \hat{\alpha}_{p,\eta^*,t}), \text{ and} \tag{21}$$

$$plim\ b_{\eta} = \frac{Cov(\Gamma_p, \hat{\alpha}_{p,\eta^*,t})}{\sigma^2_{\hat{\alpha}_{p,\eta^*,t}} + Var(\zeta_{p,\eta,t})} \tag{22}$$

Since $Var(\zeta_{p,\eta,t}) > 0$ for $\eta \neq \eta^*$, and b_{η^*} will win this horse race in a rational economy. Therefore, to determine what implications we can draw from this horse race about asset pricing models and investor sophistication, we need to identify η^* based on the parameters in the data.

1.5 Precision of alpha estimate

We first examine the sources of measurement error in alpha estimators to understand the determinants of precision. If investors know true betas they can use Eq. (6) to compute alphas, but we also examine a setting where investors estimate betas. For instance, investors may estimate betas using the following time series regression with *T* periods of data:

$$r_{p,\tau} = a_{p,t} + \sum_{k=1}^{J} \beta_{k,p,t}\, F_{k,\tau} + e_{p,\tau}, \qquad \tau = t - T \text{ to } t - 1. \tag{23}$$

The η-factor model alpha estimator using beta estimates $\hat{\beta}_{k,p,t}$ is:

$$\hat{\alpha}_{p,\eta,t} = r_{p,t} - \sum_{k=1}^{\eta} \hat{\beta}_{k,p,t}\, F_{k,t}\,. \tag{24}$$

We can substitute the true asset pricing model in Eq. (1) and the return generating process (2) in (24) to get:

$$\hat{\alpha}_{p,\eta,t} - \left(\phi_p - c(q_{t-1})\right) =$$

$$\begin{cases} \underbrace{-\sum_{k=K+1}^{\eta} \beta_{k,p} E(F_k)}_{\text{Model Misspecification error}} + \underbrace{\sum_{k=\eta+1}^{J} \beta_{k,p}\, f_{k,t} + \sum_{k=1}^{\eta} \left(\hat{\beta}_{k,p,t} - \beta_{k,p}\right) F_{k,t} + \xi_{p,t}}_{\text{Estimation error}} & \text{for } \eta \geq K, \\ \underbrace{\sum_{k=\eta+1}^{K} \beta_{k,p} E(F_k)}_{\text{Model Misspecification error}} + \underbrace{\sum_{k=\eta+1}^{J} \beta_{k,p}\, f_{k,t} + \sum_{k=1}^{\eta} \left(\hat{\beta}_{k,p,t} - \beta_{k,p}\right) F_{k,t} + \xi_{p,t}}_{\text{Estimation error}} & \text{for } \eta < K \end{cases} \tag{25}$$

ϕ_p and $c(q_{t-1})$ in this expression are unobservable but known to investors prior to time t. Therefore, these terms do not contribute to the variance of $\hat{\alpha}_{p,\eta,t}$, which is what is of interest now. Appendix 4 presents the details. The first part of the equation is model misspecification error, which we will denote by $\theta_{\eta,p}$. Model misspecification error $\theta_{\eta,p}$ is zero if $\eta = K$. If $\eta > K$, we mistakenly attribute premium for risks that are not truly priced in the economy and therefore we add noise to our alpha estimate. For example, if CAPM were the true model but we use FF3 to compute alphas, we mistakenly assume that funds with positive HML or SMB command bigger expected returns than their true expected returns. This misspecification adds to alpha estimation error.

For priced factors in an asset pricing model, stocks with bigger betas with respect to those factors earn a risk premium relative to stocks with smaller betas. But stocks' expected returns are not related to their betas with respect to unpriced factors. It is possible that $E(F_k) > 0$ for some unpriced factors. For example, Fama and French (1992) show that the cross-section of stock returns is not related to market betas and conclude that CAPM does not hold because market risk is not priced. But empirically the mean of market excess return is positive. Fama and French evidence illustrates the empirical possibility that $E(F_k)$ may be bigger than zero even if F_k is unpriced and we allow for such possibilities.[8]

The second part of the equation (25) is statistical estimation error which we denote by $\varepsilon_{\eta,p}$. Ignoring beta measurement error for now, $\varepsilon_{\eta,p} = \sum_{k=\eta+1}^{J} \beta_{k,p} f_{k,t} + \xi_{p,t}$. Therefore,

$$\sigma^2_{\varepsilon_{\eta,p}} = \sigma^2_{r_p}\left(1 - R^2_{adj,\eta,p}\right) \tag{26}$$

where $\sigma^2_{r_p}$ is the variance of fund returns and $R^2_{adj,\eta,p}$ the fraction of fund return variance that is explained by η factors with appropriate adjustment for degrees of freedom. Suppose $\varepsilon_{\eta,p}$ is uncorrelated across funds. Then statistical estimation error is the average of $\sigma^2_{\varepsilon_{\eta,p}}$ across funds. Therefore, Eq. (26) indicates that any common factor that increases R^2_{adj}, whether that factor is priced or unpriced, would reduce estimation error in $\hat{\alpha}_{p,\eta,t}$.

Measurement errors in betas would also add to statistical estimation error and affect the choice of factors that one would include in computing alphas. For instance, a factor that may marginally increase R^2_{adj} may still not be desirable if the measurement error in beta with respect to that factor increases the alpha estimation error. This issue is particularly important if that factor is correlated with other factors in the regression because the addition of that factor would increase the measurement errors of other factor betas as well.

[8] We do not take a view on whether CAPM holds based on Fama-French evidence and we use the evidence only for illustration.

There are two potential sources of measurement error in betas. Even if true betas were constant, beta estimates using a time-series would contain statistical estimation errors. Additionally, fund betas would vary over time because individual stock betas may be time-varying and active funds typically revise their portfolios over time. Therefore, the difference between the true betas at time *t+1* and the average beta during the estimation period would add to the measurement error in betas.

Eq. (25) indicates that if $\eta < K$, each factor that we omit from K adds to both misspecification error and to estimation error. Therefore, if betas with respect to a priced factor are known without error then inclusion of that factor would reduce measurement error. However, if $\eta > K$, each additional factor would reduce the estimation error, but add to misspecification error. Therefore, whether investors would optimally include these additional factors depends on the relative contribution to these components in the data, which can only be empirically determined.

1.6 CAPM vs. No-beta risk premium model: An illustrative example

This subsection considers an example that illustrates the contribution of $\sigma^2_{\varepsilon_\eta}$ and $\sigma^2_{\theta_\eta}$ to precision of the alpha estimates. Suppose asset returns are generated by the following single factor model:

$$r_{p,t} = E[r_p] + \beta_p \times f_t + \xi_{p,t}. \tag{27}$$

Expected returns are determined by one of the following two models:

i. NBRP model: The expected returns on all stocks are equal, i.e.

$$E[r_p] = E[r_m] \;\; \forall\, p \tag{28}$$

where $E[r_m]$ is the expected return on the market portfolio.

ii. CAPM:

$$E[r_p] = r_f + \beta_p (E[r_m] - r_f) \tag{29}$$

Consider the following two Estimators of alpha:

- Market adjustment (No-beta risk model):

$$\hat{\alpha}_{p,0} = r_{p,t} - r_{m,t} \tag{30}$$

- Market model adjustment (CAPM):

$$\hat{\alpha}_{p,1} = r_{p,t} - \left[r_f + \hat{\beta}_p\left(r_{m,t} - r_f\right)\right] \tag{31}$$

where $\hat{\beta}_p$ is computed using market model regression.

The variance of measurement errors of $\hat{\alpha}_{p,0}$ and $\hat{\alpha}_{p,1}$, which include both model misspecification error and statistical estimation error are tabulated below (Appendix 1 presents the derivations):

	Alpha Estimator	
True model:	a. Estimated with Mkt Adj. (Eq. 30)	b. CAPM (Eq. 31)
i. No-beta risk premium model	$\sigma_u^2 \vert r_{m,t} = \sigma_\beta^2\left(r_{m,t} - E[r_m]\right)^2 + \sigma_\xi^2 \vert r_{m,t}$	$\sigma_u^2 \vert r_{m,t} = \sigma_\beta^2 E(r_m)^2 + \sigma_{\hat{\beta}-\beta}^2 r_{m,t}^2 + \sigma_\xi^2 \vert r_{m,t}$
ii. CAPM	$\sigma_u^2 \vert r_{m,t} = \sigma_\beta^2\, r_{m,t}^2 + \sigma_\xi^2 \vert r_{m,t}$	$\sigma_u^2 \vert r_{m,t} = \sigma_{\hat{\beta}-\beta}^2 r_{m,t}^2 + \sigma_\xi^2 \vert r_{m,t}$

The variables in the table above are:

Variables	Definition
$\sigma_u^2 \vert r_{m,t}$	Variance of total measurement error conditional on the realization of market return i.e. $\sigma_u^2 \vert r_{m,t} = \sigma_{\hat{\alpha}-\alpha}^2 \vert r_{m,t}$
σ_β^2	Variance of true beta across funds.
$\sigma_{\hat{\beta}-\beta}^2$	Variance of measurement error across funds both due to the standard error of regression estimates and also due to time-variation in beta.

$\sigma_{\hat{\beta}}^2$	Variance of $\hat{\beta}_p$ across funds $= \left(\sigma_{\beta}^2 + \sigma_{\hat{\beta}-\beta}^2\right)$
$\sigma_{\xi}^2 \lvert r_{m,t}$	Variance of fund specific returns (assumed to be the same for all funds for expositional convenience) conditional on the realization of market return

The results in the above table illustrate the factors that contribute to total measurement error and the inherent trade-offs. For example, the term in cell (i)(b) can be grouped as:

$$\sigma_u^2 | r_{m,t} = \underbrace{\sigma_{\beta}^2 E[r_m]^2}_{\text{Model Misspecification Error}} + \underbrace{\sigma_{\hat{\beta}-\beta}^2 r_{m,t}^2 + \sigma_{\xi}^2 | r_{m,t}}_{\text{Estimation Error}} \tag{32}$$

The first term in this expression is variance of model misspecification error, which arises because of using market model adjustment in equation (31) when ‘no-beta risk’ model is true. The last two terms are due to statistical estimation error.

To consider the trade-offs between model misspecification error and estimation error, consider the last row where CAPM is true. The variances of estimation errors in alpha using Equations (30) and (31) are given in the last row of the table, and they both contain the term σ_{ξ}^2. The variance of alpha estimated with Equation (30) contains the additional term $\sigma_{\beta}^2 r_{m,t}^2$, which is the cross-sectional variation of true fund beta, and that with Equation (31) contains the term $\sigma_{\hat{\beta}-\beta}^2$, which is the variance of measurement error in beta. If the beta estimates are sufficiently noisy (i.e. big $\sigma_{\hat{\beta}-\beta}^2$) or if differences in betas across funds are small, then the variance of measurement error with Eq. (30) could be smaller than with estimator (31). In this case, we can infer from equation (22) that the slope coefficient b_{η} in equation (9) would be bigger for the market adjusted $\hat{\alpha}$ from estimator (30) compared to the market model $\hat{\alpha}$ from estimator (31). In other words, estimate of alpha using Eq. (30) would win out in a horse race of slope coefficients against alpha estimated with Eq. (31) even when CAPM is true (a counterexample to the underlying assumption in BvB), and even if investors were truly sophisticated (a counterexample to the underlying assumption in BHO because investors optimally do not use all factors in the return generating process). Of course, this is only an illustrative example, and we should empirically examine the true parameters to understand what we can learn from the horse races.

2. Simulation Experiment

BvB hypothesize that the winner of the alpha-fund flow horse race is the true asset pricing model, but BHO hypothesize that the winner would be the model that includes all priced and unpriced factors if investors are sophisticated. We can formally state their hypotheses as follows:

Suppose the true asset pricing model is a K-factor asset pricing model and returns are generated by a J-factor model. When we fit regression (9) with alpha computed with each η-factor model where $0 \leq \eta \leq J$, the biggest slope coefficient obtains when $\eta = \eta^*$ i.e. when $\hat{\alpha}_{p,\eta^*}$ is computed with respect to an η^*-factor model.

A1. Asset Pricing test hypothesis: The model that yields the biggest correlation is the true asset pricing model, i.e. $\eta^* = K$.

A2. Investor Sophistication hypothesis (BHO): The most accurate model is the J-factor model that generates asset returns, i.e. $\eta^* = J$.

However we show in Section 1 the winner need not necessarily be a *K* or *J* factor model because the winner broadly depends on the following factors: (i) extent to which various factor models explain fund returns (i.e. model R^2_{adj}), (ii) beta estimation error $\left(\sigma^2_{\hat{\beta}-\beta}\right)$, (iii) variation of betas across funds $\left(\sigma^2_{\beta}\right)$ and (iv) the "true" asset pricing model. Therefore, the winner would depend on the characteristics of the data, and we can only empirically identify η^*.

To do so, we can estimate the first three of the four items we list above from the data but we do not know the "true" asset pricing model. Therefore, we estimate the first three items and use these parameters to generate simulated returns under each asset pricing model. We then run the horse race with regressions (9) in the simulation to determine which factor model would win the race in a rational expectations economy, which in turn would inform us the implications we can draw from the horse race.

2.1 Data and Simulation parameters

We estimate the parameters for the simulation with the sample of funds in the CRSP survivor-bias free mutual fund database. Our sample includes all actively managed domestic equity funds in the January 1990 to June 2017 sample period. Our sample is comprised of all actively managed domestic equity funds. CRSP identifies these funds with objective codes 'EDC' and 'EDY.' When a fund has multiple share classes, we add assets in all share classes to compute its TNA and we compute fund level return as the weighted average of returns of individual share classes with lagged TNA as weights. The sample for month *t* includes all funds with at least $10 million assets under management as of the end of month *t-1*. We follow BHO and exclude funds that had flows smaller than -90% or greater than 1000% in any month from the sample to avoid the effect of outliers. The sample for month *t* includes only funds that have returns data in all months from *t*-61 to *t*-1 to estimate betas.[9]

Table 1 presents the summary statistics for the funds in our sample. The sample is comprised of 1224 funds per month on average. The average monthly fund flow into a fund is 0.25% of its TNA the previous month. Around half of the funds in the sample have either an entry or exit load.

2.2 R^2_{adj} and beta measurement error: A first look

We use the seven factor model from BHO as the *J*-factor model that generates returns. The seven factors are the three Fama-French factors (market $(mkt - r_f)$, SMB and HML), Carhart (1996) momentum factor (UMD), and three industry factors (IND_1, IND_2 and IND_3). Following BHO, we construct the three industry factors as the first three principal components of residuals from regressing Fama-French 17 equal weighted industry portfolios on FFC4 factors.

Before we proceed with the simulation, we take a first look at some of the determinants of the accuracy of alpha estimates. One important determinant is the incremental explanatory power of each additional factor. We fit the following time series regression with η factors each month t using data for each fund from months t-60 to t-1 and compute average R^2_{adj} for each model:

$$r_{p,\tau} = a_{p,\eta,t} + \sum_{k=1}^{\eta} \beta_{k,p,t} F_{k,\tau} + e_{p,\eta,\tau}, \qquad \tau = t - 60 \text{ to } t - 1. \tag{33}$$

[9] This sample selection criterion excludes funds from the sample during the first 60 months of their existence. Therefore, our sample is not exposed to potential incubation bias that Evans (2010) and Elton, Gruber and Blake (2001) document.

Table 2 reports the time-series averages. For the market model, we compute R^2_{adj} as $1 - \left(\Sigma(r_{p,t} - r_{mkt})^2 / \Sigma(r_{p,t} - \bar{r}_p)^2\right)$.

Market-adjusted returns have the lowest R^2_{adj} of .774. The R^2_{adj} for the single factor market model is bigger at .820. R^2_{adj} increase to .892 for the Fama-French three-factor model, but the increase is fairly gradual as we go from the Fama-French three-factor model to the seven factor model.

Another important component in the measurement error of $\hat{\alpha}$ is the variance of measurement error in betas across funds $\left(\sigma^2_{\hat{\beta}-\beta}\right)$. The term $\sigma^2_{\hat{\beta}-\beta}$ would differ from the time series variance of OLS estimation error in regression (9) for two reasons. First, if the fund-specific returns are correlated across funds, then the average variance of time-series errors will not equal $\sigma^2_{\hat{\beta}-\beta}$. Secondly, as we discussed earlier the OLS estimates are unbiased estimates of mean betas during the estimation periods and any difference between this average and the realized beta in month *t+1* is an additional source of measurement error.

To estimate the magnitude of this error we first estimate the following regressions for each fund for each month:

$$\begin{aligned} (r_{p,\tau} - r_{f,\tau}) &= \alpha^{past}_{p,k,t} + \beta^{past}_{p,k,t} F_{k,\tau} + e_{p,k,\tau} \qquad \tau = t - 60 \text{ to } t - 1, \\ (r_{p,\tau} - r_{f,\tau}) &= \alpha^{future}_{p,k,t} + \beta^{future}_{p,k,t} F_{k,\tau} + e_{p,k,\tau} \qquad \tau = t \text{ to } t + 11 \end{aligned} \tag{34}$$

where $F_{k,\tau}$ is the factor with respect to which betas are estimated. Suppose betas for a particular fund are constant over time.

$$\begin{aligned} \hat{\beta}^{past}_{p,k,t} &= \beta_{p,k} + u^{past}_{p,k,t}, \text{ and} \\ \hat{\beta}^{future}_{p,k,t} &= \beta_{p,k} + u^{future}_{p,k,t} \end{aligned} \tag{35}$$

where $\beta_{p,k}$ is fund *p*'s true beta with respect to factor *k*.

Consider the following cross-sectional regression for month *t*:

$$\hat{\beta}^{future}_{p,k,t} = a_t + b_t \times \hat{\beta}^{past}_{p,k,t} + e_{p,t} \tag{36}$$

Since we use non-overlapping sample periods to estimate $\beta_{p,k,t}^{past}$ and $\beta_{p,k,t}^{future}$, $u_{p,k,t}^{past}$ and $u_{p,k,t}^{future}$ are uncorrelated. With a sufficiently large number of funds, the probability limit of the slope coefficient is:

$$\text{plim}\, b_t = \frac{var(\beta_{p,k})}{var(\beta_{p,k}) + var\left(u_{p,k,t}^{past}\right)} \tag{37}$$

Therefore, the slope coefficient of regression (35) is the ratio of the cross-sectional variance of the factor betas divided by the sum of this variance plus the variance of the measurement error. If this slope coefficient is smaller than 0.5 then the variance of true beta is smaller than the variance of measurement error.

We fit regression (35) each month for each of the betas. All betas are estimated using univariate regressions as per equation (34). Table 3 reports the time-series averages of the slope coefficients for each beta. The slope coefficients are all greater than .75 for betas with respect to the three Fama-French factors, but they are less than .5 for UMD and industry factors. Therefore, the variance of measurement error is bigger than the variance of true betas for the latter set of factors.

2.3 Simulation: Experimental design

To understand how the true asset pricing model and estimation error in alphas impact the outcome of the alpha-flow horse race regressions, we simulate a mutual fund economy with parameters that match the actual sample of domestic equity funds described in section 2.1. and we match the entry and exit of funds in the simulation to that in the actual data. In this simulated economy, the fund size evolves over time with flows, net returns generated from managerial skill and net returns from passive factor exposures. And fund size affects net returns through its effect on costs.

The sample of mutual funds and their TNA evolve as follows in the simulation:

a. **Fund origin:** We start the simulation with the number of funds equal to that in the sample on January 1985.
b. **Skill:** The average four factor alpha in our actual sample of domestic equity funds, gross of fund fees F_p, is around 5 bps per month. To account for unobservable costs $C(q)/q$, we

add 10 bps per month to this estimate to account for average transaction costs[10]. In the first month when a fund enters the sample, we randomly draw ϕ_p for each fund from a normal distribution with mean equal to 0.15% and standard deviation of 0.2% per month.[11]

c. **Betas**: We randomly generate the seven factor betas for each fund from a normal distribution with means and standard deviations equal to the parameters tabulated in Table 4.[12] Each factor beta is drawn independently and is constant over the entire sample period.

d. **Fund specific return**: We generate monthly fund specific return $\epsilon_{p,t}$ for each fund from a normal distribution with mean zero and standard deviation equal to 2.5%.

e. **Asset pricing model and expected returns**: Steps (b) through (d) describe the return generating process for the funds and this process does not vary with the asset pricing model. However, different common factors that are priced vary across asset pricing models and hence different asset pricing models imply different expected return for each fund. The term $E^{model}(r_p - r_f)$ is the "true" expected excess return and it depends on the model. We conduct simulations under three asset pricing models and expected excess returns under each model are computed as follows:

- NBRP risk model: $E^{NR}(r_p - r_f) = 0.699\%$,
- CAPM: $E^{CAPM}(r_p - r_f) = \beta_{p,m} \times (\overline{mkt - r_f})$,
- Fama-French three factor model (FF3): $E^{FF3}(r_p - r_f) = -0.016\% + \beta_{p,m} \times (\overline{mkt - r_f}) + \beta_{p,smb} \times (\overline{SMB}) + \beta_{p,hml} \times (\overline{HML})$,

(38)

The overbars above common factor returns indicate sample means. The constant in the equation for each model is chosen so that the average fund returns equal sample average of market excess returns.

f. **Gross returns:** We generate fund returns using the following seven-factor model:

[10] Elton et. al. (2012) report that the transaction costs are of the same order of magnitude as expense ratios which average to around 10 bps per month.

[11] The monthly cross-sectional variance of $\hat{\alpha}$s in the real data is the variance of true alphas plus the measurement error of alphas. The measurement error variance in $\hat{\alpha}$s is the squared OLS standard errors from the time-series regressions used to estimate alphas. The average standard deviation of the difference across models is roughly 0.2% per month.

[12] As Eq. (37) shows, the standard deviation of true beta distribution in the data is the standard deviation of estimated beta multiplied by the square root of the respective slope coefficients in Table 3.

$$R_{p,t} = \phi_p + E^{model}(r_p) + \beta_{p,m} \times \left(\widetilde{mkt - r_f}\right)_t + \beta_{p,smb} \times \widetilde{SMB}_t + \beta_{p,hml} \times \widetilde{HML}_t + \beta_{p,umd} \times \widetilde{UMD}_t + \beta_{p,ind1} \times \widetilde{IND1}_t + \beta_{p,ind2} \times \widetilde{IND2}_t + \beta_{p,ind3} \times \widetilde{IND3}_t + \epsilon_{p,t} \quad (39)$$

g. **Net returns**: Net returns are given as $r_{p,t} = R_{p,t} - c(q_{t-1})$. Using the competitive equilibrium condition which requires $c(q_{t-1}) = \phi_{p,\eta,t-1}$, we compute net returns as $r_{p,t} = R_{p,t} - \phi_{p,\eta,t-1}$ where $\phi_{p,\eta,t-1}$ is the mean of investors' posterior estimate of skill ϕ_p for the period t.

h. **Mean of investors' posterior:** As the fund's age evolves, we use the following recursion to generate the mean of investors' posterior estimate of skill, which is a variant of Eq. (8) and is derived in Appendix 3:

$$\phi_{p,\eta,\tau} = \phi_{p,\eta,\tau-1} + \frac{\vartheta_{\hat{\alpha},\eta^*}}{\nu + \tau\vartheta_{\hat{\alpha},\eta^*}} \hat{\alpha}_{p,\eta^*,t}\,, \quad (40)$$

where τ indicates the age of the fund as of time t. At $\tau = 0$, we the mean of investors' prior on skill as 0.15% as mentioned in step (b) above. ν is the reciprocal of the variance of ϕ_p and $\vartheta_{\hat{\alpha},\eta^*}$ is the reciprocal of the variance of $\hat{\alpha}_{p,\eta^*,t}$ which is equal to the variance of fund specific returns specified in step (d) above when $\eta^* = 7$.

i. **Fund flow**: For each month, we compute flows using the flowing equation:

$$flow_{p,t} = a + b \times \hat{\alpha}_{p,\eta^*,t} + \psi_{p,t}. \quad (41)$$

We estimate a and b from the data, and our estimates are a = -0.00225 and b =.2, using $\eta^* = 7$. In the simulation, we draw $\psi_{p,t}$ from a normal distribution with mean zero and standard deviation of 0.09 (9%). All these parameters match the corresponding parameters in the data.[13]

j. **Fund exit and entry:** If the number of funds in the data in month *t* is smaller than the number of funds in month *t-1,* the appropriate number of funds exit the simulation sample

[13] In principle, we can use Eq. (13) to determine flows according to the model. The linear regression specification (14) is consistent with the model derived flow equation in Eq. (13) where b is determined by the average model covariance. However, the variance of $\psi_{p,t}$ from the data in (41) is smaller than the model-implied variance of the noise term. Since the winner of the horse race does not depend on the variance of this noise, we use the variance from the data.

as well. We sort funds in the simulated sample based on $\phi_{p,\eta,t}$ and drop the bottom most funds equal in number to the actual exits for that month. If the number of funds in the data in month *t* is bigger than the number of funds in month *t-1*, the appropriate number of funds enter the sample.

We repeat the simulation 100 times.

2.4 Simulation: Tests and results

We first examine the relation between alphas and fund flows under various models. We conduct two sets of tests. In the first set of tests, we use true betas and compute $\hat{\alpha}_{p,\eta,t}$ using Eq. (6). In the other set of tests, we estimate betas using the time-series regression (23) with 60 months of past data and compute alphas using Eq. (24).

We first examine the components of measurement error in compute $\hat{\alpha}_{p,\eta,t}$ using the decomposition in Eq. (25). Since $\hat{\alpha}_{\eta} \equiv \theta_{\eta} + \varepsilon_{\eta}$ where θ_{η} is the model misspecification error and ε_{η} is the statistical estimation error,

$$Var(\hat{\alpha}_{p,\eta}) = Var(\theta_{\eta}) + Var(\varepsilon_{\eta}) + Cov(\theta_{\eta}, \varepsilon_{\eta}). \qquad (42)$$

To estimate the variances in Eq. (42), we first compute the values of $\theta_{\eta}, \epsilon_{\eta}$ for different η-factors models in our simulated sample. Using these values, we compute the monthly cross-sectional variances of θ, ϵ as well as their covariance. We then average these values across time to get the required estimates.

Table 5 presents the components of alpha estimation error variance for each asset pricing model and η-factor model. Consider the results when true betas are known. Estimation error variance decreases monotonically as we increase η from zero to seven for all asset pricing models. For example, under the CAPM, estimation error variance is 875 for $\eta = 0$, which reduces to 625 for $\eta = 0$.

The model misspecification error variance increases monotonically as we move away from the true asset pricing model. However, model misspecification error variance is an order of magnitude smaller than estimation error variance. For instance, the smallest estimation error variance is 625 and in comparison the largest model misspecification error variance is 1.75, which is about 2.8% of 625.

The total error variance also monotonically declines as we increase the number of factors, which is similar to the pattern we see for the estimation error. Model misspecification error is so small in all instances that it hardly moves the needle. Therefore, if we can observe betas without error then the *J*-factor model is the optimal model, as long as each factor increases the R^2_{adj}.

When betas are not known and we estimate betas using data for 60 months, estimation error variance exhibits a U-shaped pattern. It decreases as we go from $\eta = 0$ to FF3 and then increases monotonically as we add more factors. For example, under the CAPM, estimation error variance decreases from 874.5 for $\eta = 0$ to 702.4 for $\eta = 3$, but then increases to 760.4 for $\eta = 7$. As we saw in Table 3, beta measurement error is relatively large for UMD and the three factor industry factors. Consequently, accounting for these factors to compute alpha increases estimation error variance. As before, model misspecification error is so small that it does not make difference when we compare total estimation error across models.

We next fit Regression (9) each month and estimate the coefficients and standard errors using OLS with month fixed effects. Table 6 reports the results. As we showed analytically, the magnitude of the slope coefficients across models would be negatively related to the precision of alpha estimates, and we see this pattern in Table 6. Without beta measurement error, the slope coefficient increases monotonically as we add factors. For example, under the CAPM, the slope increases from 13.93 for $\eta = 0$ to 19.97 for $\eta = 7$. The average regression R^2 also increases from .29 to .39.

The ordering of the slope coefficients across models when we estimate betas from the data is exactly the opposite of the ordering of estimation error variance in Table 5. For all asset pricing models, we find the biggest slope coefficients for $\eta = 4$. The slope coefficients for $\eta = 7$ are comparatively smaller, and the difference is statistically significant. For example, with CAPM, the slope coefficient is 17.81 for $\eta = 4$ and 16.96 for $\eta = 7$ with a statistically significant difference of 0.85.

The slope coefficients are almost identical under different true asset pricing models both when we know the true beta and when we estimate beta from the data. These results indicate that the relation between alpha and flow is not particularly sensitive to the true asset pricing model. Therefore, one cannot identify the true asset pricing model using the alpha-flow horse race.

3. Binary variable regression

Our analysis in the last section uses a linear regression for the alpha-fund flow horse race. However, Berk and Green (2004) show that the equilibrium relation between alpha and fund flows is nonlinear. Because of the non-linearity, BvB transform flows and alpha estimates to binary variables and run the horse race with these transformed variables. Specifically, the transformed binary variables are defined as follows:

$$Q_x = \begin{cases} 1 \text{ if } x \geq 0 \\ -1 \text{ if } x < 0 \end{cases}, \tag{43}$$

where x is any random variable. BvB run the following OLS regression:

$$Q_{\Gamma_p} = A_\eta + B_\eta \times Q_{\hat{\alpha}_{p,\eta}} + o_{p,\eta}, \tag{44}$$

and compare $\hat{B}_\eta$. To relate our analysis based on Regression (9) to that based on Regression (44), we first establish the following propositions:

Proposition 3: Let $\hat{\alpha}_{p,\eta_1}$ and $\hat{\alpha}_{p,\eta_2}$ be the alphas computed with respect to η_1- and η_2-factor models using Equation (6). $\hat{b}_{\eta_1}$ and $\hat{b}_{\eta_2}$ are the corresponding Regression (9) slope coefficients and $\hat{B}_{\eta_1}$ and $\hat{B}_{\eta_2}$ are the corresponding Regression (44) slope coefficients. Under the augmented Berk and Green model,

If $\hat{b}_{\eta_1} > \hat{b}_{\eta_2}$ then $\hat{B}_{\eta_1} > \hat{B}_{\eta_2}$, when the number of funds in the sample is sufficiently large.

Proof: See Appendix 2.

Corollary: The ordering of the slope coefficients of Regressions (9) and (44) are identical.

Proposition 3 and its corollary show that our analysis of the horse race based on Regression (9) applies exactly to that of the horse race based on Regression (44).

4. Results in Perspective

BvB, BHO, Agarwal, Green and Ren (2017) and Blocher and Molyboga (2017) report that single factor alpha is most highly correlated with fund flows into mutual funds and hedge funds among alphas computed with respect to many multifactor models. BvB and some other papers conclude that these results indicate that the CAPM is the true asset pricing model. However, BHO conclude that these results indicate that investors lack the sophistication to use the most precise model to estimate alphas for their investment decision. What are the assumptions that are necessary to draw these inferences? Are these assumptions satisfied in the data?

Our analysis shows that any inference about the true asset pricing model is tenable only if inclusion of any of the unpriced factors to compute alphas in Eq. (6) increases alpha estimate variance due to model misspecification error more than it reduces statistical estimation error. BvB effectively make such an assumption when they assume "if a true risk model exists, any false risk model cannot have additional explanatory power." BvB note that this assumption "rules out the possibility that ε_{it}^{c} contains information about managerial ability that is not also contained in ε_{it}" where their notations ε_{it} and ε_{it}^{c} denote alpha estimation errors with the true asset pricing model (i.e. $\hat{\alpha}_{p,K}$ estimated using the *K*-factor model) and with any other multifactor model $\left(\text{i.e. } \hat{\alpha}_{p,\eta}\ \forall\, \eta \neq K\right)$, respectively.

Is this assumption empirically tenable? Our simulation shows for the parameters in the data, the precision of alpha estimate is insensitive to the true asset pricing model. For example, if CAPM were the true model but we estimate alphas using the seven-factor model, the increase in model misspecification error is an order of magnitude smaller than the decrease in estimation error compared with the error in $\hat{\alpha}_{p,K}$. In fact, the winner of the horse race does not depend on the true asset pricing model both if we know the true betas and if we estimated betas from the data. Therefore, any inference about the true asset pricing model based on alpha-fund flow horse race is empirically untenable.

Regarding inferences about investor sophistication, an important question is, what is the appropriate benchmark that sophisticated investors would use? BHO hypothesize that sophisticated investors would use the *J*-factor model, a model that includes all priced and unpriced common factors. Our analysis shows that this hypothesis ignores the potential contribution of model misspecification error and the effect of measurement error in factor betas.

Our simulation results indicate that *J*-factor model indeed wins out horse race regardless of the true asset pricing model if betas are known. However, when we estimate betas with 60 months of data, alphas are estimated more precisely with the three factor model than with the seven factor model. Therefore, in this case the appropriate benchmark for assessing investor sophistication is the four factor model rather than the seven factor model. The evidence in BHO that market model alphas win the horse race indicates that investors use this alpha to inform their investment decision rather than the most precise three factor alpha. Therefore, their conclusion that investors are not sophisticated enough to use the most precise estimate of alpha to inform their mutual fund investment decisions is empirically tenable.

5. **Conclusion**

Investors reveal their preferences for mutual funds through investments in or withdrawals from them. Since non-satiated investors prefer more abnormal returns to less, investors' fund flows reveal their views on abnormal returns that they can earn from their investments. Because flows reveal investors' perceptions, the recent literature has proposed that a comparison of relations between fund flows and alphas measured with respect to a number of models can be used to identify the best asset pricing model and also to assess investor sophistication.

We show analytically that the empirical tenability of any inferences we draw based on such flow-alpha horse race critically depend on the sources of measurement error in alphas estimated under various models. For instance, we show that we can draw reliable inferences about asset pricing models only if the dominant source of error in alphas is due to the misspecification of the true asset pricing model. However, we find that the true asset pricing model has no effect on the ordering of the flow-alpha relations in our simulations with parameters estimated form the data. These findings indicate that asset pricing model misspecification error is a trivial of alpha estimation error in the data. Therefore, any inference about the true asset pricing model based on the flow-alpha horse race is empirically untenable.

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Table 1: Summary statistics

This table presents the summary statistics for the sample of funds included in the sample. The number of fund-month observations is 404,042. The table first computes the respective statistics across funds each month and reports the averages over the entire sample period. The sample period is from January, 1990 to June, 2017.

	Mean	Std. Dev.	Median
Number of funds each month	1224		
Flow (%)	0.25	10.8	-0.42
TNA ($ mn)	1120.4	4507.4	223.6
Age (months)	376.8	306.6	299.2
Expense Ratio (%)	1.22	0.45	1.19
Load Dummy	0.49	0.50	0
Ret. Volatility (t-1,t-12)	4.7	2.3	4.2

Table 2: Factor model R^2

This table fits the following regression:

$$(r_{p,t} - r_{f,t}) = \alpha_{p,\eta} + \sum_{k=1}^{\eta} \beta_{k,p} F_{k,t} + e_{p,\eta,t}.$$

Where $r_{p,t}, r_{f,t}$ and $F_{k,t}$ are fund return, risk-free rate and realization of factor k in month t, respectively. For each month t, the regression is fitted from $t - 60$ to $t - 1$. The table reports the cross-sectional averages of time-series means of adjusted R^2 of the OLS regressions under each model. For market-adjusted and benchmark-adjusted returns we compute this metric as $1 - (\sum(r_{it} - r_{mkt})^2/\sum(r_{it} - \bar{r}_i)^2)$, $1 - \left(\sum(r_{it} - r_{b/m})^2/\sum(r_{it} - \bar{r}_i)^2\right)$ using full sample of returns for each. Benchmark is the fund benchmark identified by Cremers and Petajisto (2009). The sample period is January, 1990 to June, 2017.

Model	Adj. R^2	
	Regression	$1 - \left(Var(\alpha_{p,\eta}) / Var(r_p) \right)$
Market Adj. Return	0.774	0.761
Benchmark Adj. Return	0.870	0.824
Market Model	0.820	0.829
FF3	0.892	0.883
FFC4	0.901	0.883
FFC4 + 3 IND	0.910	0.884

Table 3: Measurement Errors in betas

This table reports the slope coefficients from the following cross-sectional regressions:

$$\hat{\beta}_{p,k,t}^{future} = a_t + b_t \times \hat{\beta}_{p,k,t}^{past} + e_{p,t},$$

where for each fund f, $\hat{\beta}_{p,k,t}^{future}$ and $\hat{\beta}_{p,k,t}^{past}$ are estimated using time-series regressions with data from t to t+11, and t-1 to t-60, respectively. All betas are estimated with univariate time-series regressions. The above regression is fitted each month for betas with respect to each factor and the table reports time-series averages of the slope coefficients. Standard errors from the second stage of Fama-MacBeth regressions are adjusted for serial correlation using Newey-West correction with lag length of 11 months. Sample period for these regressions is Jan-1990 to Jul-2016. ***, **, * indicate statistical significance at the 1%, 5%, and 10% levels respectively.

Betas	Average b_t	Std. Err.
Market	0.821***	0.07
SMB	0.876***	0.03
HML	0.765***	0.05
UMD	0.409***	0.08
IND1	0.356***	0.09
IND2	0.362***	0.09
IND3	0.090	0.10

Table 4: Simulation Parameters

This table shows the parameters used in generating simulated returns and flows during 1990-2017. We generate net returns each month using the following seven-factor model:

$$r_{p,t} = \phi_p - \phi_{p,\eta,t-1} + E^{model}(r_p) + \beta_{p,m} \times (\widetilde{mkt - rf})_t + \beta_{p,smb} \times \widetilde{SMB}_t + \beta_{p,hml} \times \widetilde{HML}_t$$
$$+ \beta_{p,umd} \times \widetilde{UMD}_t + \beta_{p,ind1} \times \widetilde{IND1}_t + \beta_{p,ind2} \times \widetilde{IND2}_t + \beta_{p,ind3} \times \widetilde{IND3}_t$$
$$+ \epsilon_{p,t}$$

where ϕ_p is the fund manager skill, $\phi_{p,\eta,t-1}$ is the mean of investors' posterior estimate of skill ϕ_p for the period t. Based on a fund's age τ as of time period t, we use the recursion $\phi_{p,\eta,\tau} = \phi_{p,\eta,\tau-1} + \hat{\alpha}_{p,\eta^*,t}\left(\vartheta_{\hat{\alpha},\eta^*}/(\nu + \tau\vartheta_{\hat{\alpha},\eta^*})\right)$ to update investors' posterior mean of skill. At $\tau = 0$, we set the mean of investors' prior on skill ϕ_0 to be same as the mean of ϕ_p. The variables under $tilde$ are demeaned realizations of the following factors: market, SMB, HML, UMD, and three industry factors and βs are the corresponding factor sensitivities. We generate monthly flow as:

$$flow_{p,t} = a + b \times \hat{\alpha}_{p,\eta^*,t} + \psi_{p,t},$$

where $\hat{\alpha}_{p,\eta^*,t}$ is computed using the above seven factors as $r_{p,t} - \sum_{k=1}^{7} \beta_{p,k} \times F_{k,t}$. All randomly drawn parameters are generated from a Normal distribution with means and standard deviations shown in the table.

Panel A: Randomly drawn parameters		
Parameter	Mean	Standard Deviation
ϕ_p	ϕ_0 =0.15%	$1/\sqrt{\nu}$=0.2%
β_{mkt}	1	0.154
β_{smb}	0.25	0.328
β_{hml}	0	0.262
β_{umd}	0	0.096
β_{IND1}	0	0.036
β_{IND2}	0	0.036
β_{IND3}	0	0.024
ϵ	0	$1/\sqrt{\vartheta_{\hat{\alpha},\eta^*}}$=0.025 (2.5%)
ψ	0	0.09 (9%)
Panel B: Fixed parameters		
Parameter	Value	
a	-0.00225	
b	0.2	
η^*	7 factor model	

Table 5: Measurement error components in simulated sample

This table shows the empirical estimates of various components in the variance decomposition of measurement error in $\hat{\alpha}$ from Eq (32). Each month, in the simulated sample, we compute the model misspecification error (θ) and statistical estimation error (ϵ) for various combinations of true asset pricing models (K=0, 1, 3) and estimation models (η=0, 1, 3, 4, 7) using the analytical expressions from Eq. (31). From these values, we compute the monthly cross-sectional variances and covariance and then average them across time and across 50 simulation samples and report the values scaled by 10^{-6}. Columns (1), (2), (3) of each Panel in the table below show the variances of estimation error, misspecification error and the covariance of the two respectively. Panel A shows the variance estimates when we use true betas of the funds to compute $\hat{\alpha}_\eta$ in which case the beta measurement error part drops in the estimation error component. And panel B shows the variance estimates where we use 60 month rolling window estimates of $\hat{\beta}$ to compute $\hat{\alpha}_\eta$ in which case the measurement error in betas shows up as part of column (1). The expressions we use to compute θ, ϵ are:

$$\theta = -\sum_{k=K+1}^{\eta} \beta_{k,p} E(F_k) \text{ if } \eta \geq K,\ \theta = \sum_{k=\eta+1}^{K} \beta_{k,p} E(F_k) \text{ if } \eta < K \text{ and } \epsilon = \sum_{k=\eta+1}^{J} \beta_{k,p} f_{k,t} + \sum_{k=1}^{\eta} (\hat{\beta}_{k,p,t} - \beta_{k,p}) F_{k,t} + \xi_{p,t}.$$

Betas used to estimate alphas are:		Panel A: True Betas				Panel B: $\hat{\beta}$s from 60 month rolling regressions			
		σ_ϵ^2 (1)	σ_θ^2 (2)	$Cov(\theta,\epsilon)$ (3)	(1)+(2)+(3)	σ_ϵ^2 (1)	σ_θ^2 (2)	$Cov(\theta,\epsilon)$ (3)	(1)+(2)+(3)
True asset pricing Model (K):	Alpha Estimated Using (η):								
No-beta risk premium model (K=0)	Mkt adj. ret.	875.2	0	0	875.2	874.5	0	0	874.5
	Market model	832.3	1.151	0.005	833.5	833.3	1.148	0.005	834.5
	FF3	661.4	1.494	0.003	662.9	702.4	1.49	0.005	703.9
	FFC4	640.2	1.743	-0.001	642.0	717.0	1.739	-0.001	718.7
	FFC4+3 IND	625.0	1.755	-0.001	626.7	760.9	1.751	0.007	762.7
CAPM (K=1)	Mkt adj. ret.	875.2	1.151	-0.110	876.3	874.5	1.148	-0.103	875.6
	Market model	832.3	0	0	832.3	833.3	0	0	833.3
	FF3	661.4	0.350	-0.002	661.8	702.4	0.35	-0.006	702.8
	FFC4	640.2	0.602	-0.005	640.8	717.0	0.601	-0.010	717.6

	FFC4+3 IND	625.0	0.614	-0.004	625.6	760.9	0.613	-0.008	761.6
FF3 (*K=3*)	Mkt adj. ret.	875.2	1.494	-0.052	876.7	874.5	1.49	-0.039	876.0
	Market model	832.3	0.350	0.058	832.7	833.3	0.35	-0.004	833.7
	FF3	661.4	0	0	661.4	702.4	0	0	702.4
	FFC4	640.2	0.252	-0.004	640.5	717.0	0.252	-0.005	717.2
	FFC4+3 IND	625.0	0.264	-0.002	625.3	760.9	0.263	-0.002	761.2

Table 6: Flow-Performance relation in simulated sample

This table presents univariate flow-performance regression results in the simulated sample. Columns (1), (2) and (3) in each of panels A and B report the results with true expected returns generated under No-beta risk premium (NBRP), CAPM, and FF3 models. The alphas which are the independent variables are computed with respect to the models indicated in the first column. Panel A shows the results using true betas to compute these alphas while panel B shows the results with $\hat{\beta}$s estimated using the prior month returns. Monthly flow is simulated in the sample as $flow_{p.t} = -0.00225 + 0.2 * \hat{\alpha}_{p,\eta^*=7,t} + \psi_{p,t}$ which is the dependent variable. The table presents the average value of slope coefficients multiplied by 100 with flows as the dependent variable and alphas as independent variables across 50 simulated samples.

	Panel A: True betas used to estimate alphas						Panel B: 60 month rolling window $\hat{\beta}$s used to estimate alphas					
	(1)		(2)		(3)		(1)		(2)		(3)	
True asset pricing model (K):	NBRP model		CAPM		FF3		NBRP model		CAPM		FF3	
	Coef/SE	R^2	Coef/SE	R^2	Coef/SE	R^2	Coef/SE	R^2	Coef/SE	R^2	Coef/SE	R^2
Alpha Estimated Using (η):												
Market Adjusted Ret	13.949***	0.294	13.935***	0.293	13.929***	0.293	14.517***	0.339	14.499***	0.339	14.494***	0.339
	(0.445)		(0.442)		(0.440)		(0.624)		(0.623)		(0.622)	
Market model	14.691***	0.305	14.686***	0.305	14.679***	0.305	15.258***	0.352	15.252***	0.351	15.249***	0.351
	(0.462)		(0.460)		(0.458)		(0.674)		(0.673)		(0.673)	
FF3	18.753***	0.370	18.753***	0.369	18.753***	0.369	17.793***	0.392	17.790***	0.392	17.791***	0.391
	(0.541)		(0.539)		(0.535)		(0.694)		(0.694)		(0.693)	
FFC4	19.454***	0.380	19.456***	0.380	19.456***	0.380	17.810***	0.392	17.807***	0.392	17.808***	0.392
	(0.589)		(0.588)		(0.584)		(0.688)		(0.688)		(0.687)	
FFC4+3 IND	19.970***	0.388	19.973***	0.388	19.973***	0.388	16.963***	0.379	16.959***	0.378	16.958***	0.378
	(0.603)		(0.601)		(0.597)		(0.670)		(0.670)		(0.669)	
Coefficient Difference Test												
FFC4 - (FFC4+3 IND)	-0.516***		-0.517***		-0.517***		0.847***		0.849***		0.849***	
	(0.082)		(0.082)		(0.083)		(0.171)		(0.171)		(0.171)	
FFC4 - True Asset Pricing Model	5.505***		4.770***		0.703***		3.294***		2.556***		0.017	
	(0.302)		(0.278)		(0.116)		(0.335)		(0.315)		(0.144)	

Appendix 1:

This appendix derives the results presented in section 1.4 of the paper. For expositional convenience, we set the risk-free rate to zero.

Let returns be generated by a single factor model as shown in equation (16). The true model of expected returns is either a no-beta risk premium model in equation (17) or CAPM in equation (18). $\hat{\alpha}$ is estimated using either a market adjustment as shown in equation (19) or a market model adjustment as shown in equation (20).

In the cross-section of funds, the following hold true:

$$\begin{aligned} cov(\beta,\xi) &= 0 \\ cov(\hat{\beta},\xi) &= 0 \\ cov(\hat{\beta}-\beta,\xi) &= 0, \end{aligned} \qquad (A.1.1)$$

where β represents true beta of a fund, $\hat{\beta}$ represents the estimated beta of the fund, $\hat{\beta}-\beta$ is the measurement error in estimated beta, and ξ represents the fund specific returns.

We also have, by definition:

$$cov(\hat{\beta}-\beta,\beta) = 0 \qquad (A.1.2)$$

From the two models of expected returns and two estimators, we have the following four cases.

Case 1: Market adjustment when the no-beta risk model is true

From equations (16), (17), (19):

$$\begin{aligned} \hat{\alpha}_{p,0} = r_{p,t} - r_{m,t} &= \alpha_p + E[r_m] + \beta_p \times f_t + \xi_{p,t} - r_{m,t} \\ &= \alpha_p + E[r_m] + \beta_p \times (r_{m,t} - E[r_m]) + \xi_{p,t} - r_{m,t} \\ &= \alpha_p + u_t \text{ where } u_t = (\beta_p - 1) \times (r_{m,t} - E[r_m]) + \xi_{p,t} \end{aligned}$$

Therefore, the cross-sectional variance of u_t after using the results in $(A.1.1)$ will be:

$$\begin{aligned} \sigma_u^2 \Big| r_{m,t} &= \left(r_{m,t} - E[r_m]\right)^2 \times var\left(\beta_p - 1 \middle| r_{m,t}\right) + \sigma^2_{\xi_{p,t}} \Big| r_{m,t} \\ &= \left(r_{m,t} - E[r_m]\right)^2 \times \sigma^2_{\beta_p} | r_{m,t} + \sigma^2_{\xi_{p,t}} | r_{m,t} \end{aligned}$$

Since the true betas and the fund specific returns are drawn from identical distributions across funds, we can drop the subscript p to arrive at:

$$\sigma_u^2|r_{m,t} = \left(r_{m,t} - E[r_m]\right)^2 \times \sigma_\beta^2|r_{m,t} + \sigma_\xi^2|r_{m,t} \qquad (A.1.3)$$

Case 2: Market model adjustment (i.e. CAPM) when the no-beta risk model is true

From equation (20):

$$\begin{aligned}\hat{\alpha}_{p,1} &= r_{p,t} - \hat{\beta}_p \times r_{m,t} - \left(1 - \hat{\beta}_p\right) \times r_f \\ &= \alpha_p + E[r_m] + \beta_p \times f_t + \xi_{p,t} - \hat{\beta}_p r_{m,t} - \left(1 - \hat{\beta}_p\right) \times r_f \text{ from equations (13), (14)} \\ &= \alpha_p + E[r_m] + \beta_p \times \left(r_{m,t} - E[r_m]\right) + \xi_{p,t} - \hat{\beta}_p r_{m,t} - \left(1 - \hat{\beta}_p\right) \times r_f \\ &= \alpha_p + u_t \text{ where } u_t = \left(1 - \beta_p\right) \times E(r_m) - \left(\hat{\beta}_p - \beta_p\right) \times r_{m,t} - \left(1 - \hat{\beta}_p\right) \times r_f + \xi_{p,t}\end{aligned}$$

Using $(A.1.1), (A.1.2)$, and the following two results

$$Cov\left(1 - \beta_p, 1 - \hat{\beta}_p\right) = var\left(\beta_p\right)$$
$$Cov\left(\hat{\beta}_p - \beta_p, 1 - \hat{\beta}_p\right) = -var\left(\hat{\beta}_p - \beta_p\right),$$

the cross-sectional variance of u_t will be:

$$\begin{aligned}\sigma_u^2\big|r_{m,t} = E(r_m) \times \left(E(r_m) - r_f\right) \times \sigma_\beta^2\big|r_{m,t} + r_{m,t} \times (r_{m,t} - r_f) \\ \times \sigma_{\hat{\beta}-\beta}^2\Big|r_{m,t} + r_f^2 \times \sigma_{\hat{\beta}}^2\Big|\, r_{m,t} + \sigma_\xi^2|r_{m,t}\end{aligned} \qquad (A.1.4)$$

When the risk-free rate is set to zero:

$$\sigma_u^2\big|r_{m,t} = E(r_m)^2 \times \sigma_\beta^2\big|r_{m,t} + r_{m,t}^2 \times \sigma_{\hat{\beta}-\beta}^2|r_{m,t} + \sigma_\xi^2|r_{m,t} \qquad (A.1.5)$$

Case 3: Market adjustment when CAPM is true

From equations (16), (18), (19):

$$\begin{aligned}\hat{\alpha}_{p,0} &= \alpha + r_f + \beta_p \times \left(E[r_m] - r_f\right) + \beta_p \times f_t + \xi_{p,t} - r_{m,t} \\ &= \alpha + r_f + \beta_p \times \left(E[r_m] - r_f\right) + \beta_p \times \left(r_{m,t} - E[r_m]\right) + \xi_{p,t} - r_{m,t} \\ &= \alpha + u_t \text{ where } u_t = -\left(1 - \beta_p\right) \times (r_{m,t} - r_f) + \xi_{p,t}\end{aligned}$$

Using $(A.1.1)$, the cross-sectional variance of u_t is:

$$\sigma_u^2 \Big| r_{m,t} = \left(r_{m,t} - r_f\right)^2 \times \sigma_{\beta_p}^2 \Big| r_{m,t} + \sigma_{\xi_{p,t}}^2 | r_{m,t} \qquad (A.1.6)$$

Dropping the subscript p since betas and fund specific returns are drawn from identical distributions across funds and with risk free rate set to zero, we get:

$$\sigma_u^2 | r_{m,t} = r_{m,t}^2 \times \sigma_\beta^2 | r_{m,t} + \sigma_\xi^2 | r_{m,t} \qquad (A.1.7)$$

Case 4: Market model adjustment when CAPM is true

From (16), (18), (20):

$$\begin{aligned}\hat{\alpha}_{p,1} &= \alpha + r_f + \beta_p \times \left(E[r_m] - r_f\right) + \beta_p \times f_t + \xi_{p,t} - \left[r_f + \hat{\beta}_p\left(r_{m,t} - r_f\right)\right] \\ &= \alpha + r_f + \beta_p \times \left(E[r_m] - r_f\right) + \beta_p \times \left(r_{m,t} - E[r_m]\right) + \xi_{p,t} - \left[r_f + \hat{\beta}_p\left(r_{m,t} - r_f\right)\right] \\ &= \alpha + u_t \text{ where } u_t = -\left(\hat{\beta}_p - \beta_p\right) \times (r_{m,t} - r_f) + \xi_{p,t}\end{aligned}$$

Using $(A.1.1)$, the cross-sectional variance of u_t is:

$$\sigma_u^2 \Big| r_{m,t} = \left(r_{m,t} - r_f\right)^2 \times \sigma_{\hat{\beta}_p - \beta_p}^2 \Big| r_{m,t} + \sigma_{\xi_{p,t}}^2 | r_{m,t} \qquad (A.1.8)$$

After dropping subscript p and setting risk free rate to zero, we get:

$$\sigma_u^2 \Big| r_{m,t} = r_{m,t}^2 \times \sigma_{\hat{\beta} - \beta}^2 \Big| r_{m,t} + \sigma_\xi^2 | r_{m,t} \qquad (A.1.9)$$

Appendix 2:

This appendix proves Proposition 3.

Denote

$$\hat{\alpha}_{p,\eta} = \hat{\alpha}_{p,\eta^*} + v_{p,\eta}, \qquad (A.2.1)$$

where $\hat{\alpha}_{p,\eta^*}$ is the alpha estimated using the most optimal η^*-factor model. Under the rational expectations equilibrium of Berk, Green (2004), flow positively covaries with $\hat{\alpha}_{p,\eta^*,t}$ and is uncorrelated with the noise term $v_{p,\eta}$.

Under the model of Berk, Green (2004), $\hat{\alpha}_{p,\eta 1}, \hat{\alpha}_{p,\eta 2}$ are normally distributed with mean zero and are therefore symmetric around zero. Therefore:

$$\Pr\left(Q_{p,\eta} = -1\right) = \Pr\left(Q_{p,\eta} = 1\right) = .5 \text{ for } \eta = \eta_1, \eta_2,$$

$$E\left(Q_{p,\eta_1}\right) = E\left(Q_{p,\eta_2}\right) = 0 \text{, and} \qquad (A.2.2)$$

$$Var\left(Q_{p,\eta_1}\right) = Var\left(Q_{p,\eta_2}\right) = 1.$$

It also follows from the definition in $(A.2.1)$ that:

$$E\left(v_{p,\eta}\right) = 0 \qquad (A.2.3)$$

Consider the following OLS regressions from (11) and (34):

$$\Gamma_p = a_\eta + b_\eta \hat{\alpha}_{p,\eta} + \omega_{p,\eta}$$
$$Q_{\Gamma_p} = A_\eta + B_\eta Q_{p,\eta} + o_{p,\eta}$$

From Regression (11), after using $Cov\left(\Gamma_p, v_{p,\eta}\right) = 0$, we get:

$$b_\eta = \frac{cov(\Gamma_p, \hat{\alpha}_{p,\eta})}{var\left(\hat{\alpha}_{p,\eta}\right)} = \frac{cov(\Gamma_p, \hat{\alpha}_{p,\eta^*})}{var\left(\hat{\alpha}_{p,\eta^*}\right) + var(v_{p,\eta})} \qquad (A.2.4)$$

Given that $\hat{b}_{\eta_1} > \hat{b}_{\eta_2}$. Therefore, from $(A.2.4)$ we get:

$$var\left(v_{p,\eta 1}\right) < var(v_{p,\eta 2}) \qquad (A.2.5)$$

From Regression (34), after using the result in $(A.2.2)$, we get:

$$B_\eta = \frac{Cov\left(Q_{\Gamma_p}, Q_{p,\eta}\right)}{Var(Q_{p,\eta})} = \; Cov\left(Q_{\Gamma_p}, Q_{p,\eta}\right) \qquad (A.2.6)$$

To evaluate this covariance term, we use the law of total covariance which states:

$$cov(X,Y) = E\big(cov(X,Y|Z)\big) + cov(E(X|Z), E(Y|Z)) \qquad (A.2.7)$$

Using $(A.2.7)$, we can write:

$$Cov\left(Q_{\Gamma_p}, Q_{p,\eta}\right) = E\left(cov\left(Q_{\Gamma_p}, Q_{p,\eta}|Q_{p,\eta^*}\right)\right) + cov\left(E\left(Q_{\Gamma_p}|Q_{p,\eta^*}\right), E\big(Q_{p,\eta}|Q_{p,\eta^*}\big)\right) \qquad (A.2.8)$$

Since Γ_p is independent of the noise part of $\hat{\alpha}_{p,\eta}$, the conditional covariance $cov\left(Q_{\Gamma_p}, Q_{p,\eta}|Q_{p,\eta^*}\right)$ will be zero on average. Hence the first term on the RHS of $(A.2.8)$ will be zero. Expanding the second term in $(A.2.8)$, we get:

$$Cov\left(Q_{\Gamma_p}, Q_{p,\eta}\right) = E\left[E\left(Q_{\Gamma_p}|Q_{p,\eta^*}\right) \times E\big(Q_{p,\eta}|Q_{p,\eta^*}\big)\right] - E\left[E\left(Q_{\Gamma_p}|Q_{p,\eta^*}\right)\right] \times E\big[E\big(Q_{p,\eta}|Q_{p,\eta^*}\big)\big] \qquad (A.2.9)$$

The two terms in $(A.2.9)$ can further be expanded as:

$E\left[E\left(Q_{\Gamma_p}|Q_{p,\eta^*}\right) \times E\big(Q_{p,\eta}|Q_{p,\eta^*}\big)\right] = E\left(Q_{\Gamma_p}|Q_{p,\eta^*} = 1\right) \times E\big(Q_{p,\eta}|Q_{p,\eta^*} = 1\big) \times \Pr\big(Q_{p,\eta^*} = 1\big) + E\left(Q_{\Gamma_p}|Q_{p,\eta^*} = -1\right) \times E\big(Q_{p,\eta}|Q_{p,\eta^*} = -1\big) \times \Pr\big(Q_{p,\eta^*} = -1\big)$, and

$E\left[E\left(Q_{\Gamma_p}|Q_{p,\eta^*}\right)\right] \times E\big[E\big(Q_{p,\eta}|Q_{p,\eta^*}\big)\big] = \left\{E\left(Q_{\Gamma_p}|Q_{p,\eta^*} = 1\right) \times \Pr\big(Q_{p,\eta^*} = 1\big) + E\left(Q_{\Gamma_p}|Q_{p,\eta^*} = -1\right) \times \Pr\big(Q_{p,\eta^*} = -1\big)\right\} \times \left\{E\big(Q_{p,\eta}|Q_{p,\eta^*} = 1\big) \times \Pr\big(Q_{p,\eta^*} = 1\big) + E\big(Q_{p,\eta}|Q_{p,\eta^*} = -1\big) \times \Pr\big(Q_{p,\eta^*} = -1\big)\right\}$

Substituting these into $(A.2.9)$, using $1 - \Pr(Q_{p,\eta^*} = 1) = \Pr(Q_{p,\eta^*} = -1)$, and rearranging the terms yields:

$$
\begin{aligned}
&Cov\left(Q_{\Gamma_p}, Q_{p,\eta}\right) \\
&\qquad = \left\{E\left(Q_{\Gamma_p}|Q_{p,\eta^*} = 1\right) - E\left(Q_{\Gamma_p}|Q_{p,\eta^*} = -1\right)\right\} \\
&\qquad \times \left\{E(Q_{p,\eta}|Q_{p,\eta^*} = 1) - E(Q_{p,\eta}|Q_{p,\eta^*} = -1)\right\} \times \Pr(Q_{p,\eta^*} = 1) \\
&\qquad \times \Pr(Q_{p,\eta^*} = -1)
\end{aligned}
\tag{A.2.10}
$$

From $(A.2.6)$ and $(A.2.10)$ we can see that comparing coefficients $B_{\eta 1}, B_{\eta 2}$ reduces to comparing $\{E(Q_{p,\eta}|Q_{p,\eta^*} = 1) - E(Q_{p,\eta}|Q_{p,\eta^*} = -1)\}$ for $\eta = \eta_1$ & η_2, since η^* is same across the two models.

By definition, this term can be expressed as:

$$
\begin{aligned}
&E(Q_{p,\eta}|Q_{p,\eta^*} = 1) - E(Q_{p,\eta}|Q_{p,\eta^*} = -1) \\
&\qquad = \Pr(\hat{\alpha}_{p,\eta^*} + v_{p,\eta} \geq 0|\hat{\alpha}_{p,\eta^*} \geq 0) \\
&\qquad - \Pr(\hat{\alpha}_{p,\eta^*} + v_{p,\eta} < 0|\hat{\alpha}_{p,\eta^*} \geq 0) \\
&\qquad - \Pr(\hat{\alpha}_{p,\eta^*} + v_{p,\eta} \geq 0|\hat{\alpha}_{p,\eta^*} < 0) \\
&\qquad + \Pr(\hat{\alpha}_{p,\eta^*} + v_{p,\eta} < 0|\hat{\alpha}_{p,\eta^*} < 0)
\end{aligned}
\tag{A.2.11}
$$

Where the conditional probabilities are defined as:

$$
\begin{aligned}
&\Pr(\hat{\alpha}_{p,\eta^*} + v_{p,\eta} \geq 0|\hat{\alpha}_{p,\eta^*} \geq 0) \\
&\qquad = \int_0^\infty \Pr(\hat{\alpha}_{p,\eta^*} \geq -v_{p,\eta}|\hat{\alpha}_{p,\eta^*}) \times f(\hat{\alpha}_{p,\eta^*}|\hat{\alpha}_{p,\eta^*} \geq 0) \times d\hat{\alpha}_{p,\eta^*}
\end{aligned}
\tag{A.2.12}
$$

with $v_{p,\eta}|\hat{\alpha}_{p,\eta^*}$ distributed as Normal with mean zero.

We get similar expressions for the remaining three terms on the RHS of equation $(A.2.11)$.

When $X \sim N(0, \sigma^2)$, the following definitions apply:

$$
\begin{aligned}
\Pr(X \leq a) = F(a) &= \frac{1}{2} \times \left[1 + \operatorname{erf}\left(\frac{a-\mu}{\sigma\sqrt{2}}\right)\right] = \frac{1}{2} \times \left[1 + \operatorname{erf}\left(\frac{a}{\sigma\sqrt{2}}\right)\right] \\
\Pr(X \geq a) = 1 - F(a) &= \frac{1}{2} \times \left[1 - \operatorname{erf}\left(\frac{a-\mu}{\sigma\sqrt{2}}\right)\right] = \frac{1}{2} \times \left[1 - \operatorname{erf}\left(\frac{a}{\sigma\sqrt{2}}\right)\right]
\end{aligned}
\tag{A.2.13}
$$

Where $\text{erf}(x)$ is the error function given by:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \times \int_0^x e^{-t^2} dt$$

This is an odd function with $\text{erf}(-x) = -\text{erf}(x)$ and is monotonically increasing in its argument x. From these two properties and the definitions in $(A.2.13)$, we can infer the following:

$$\begin{aligned} \Pr(X \geq a) \ \text{ is } &\begin{cases} \text{decreasing with } \sigma \text{ if } a < 0 \\ \text{increasing with } \sigma \text{ if } a > 0 \end{cases} \\ \Pr(X \leq a) \ \text{ is } &\begin{cases} \text{increasing with } \sigma \text{ if } a < 0 \\ \text{decreasing with } \sigma \text{ if } a > 0 \end{cases} \end{aligned} \qquad (A.2.14)$$

From $(A.2.5)$, we have $\sigma_{v_{p,\eta 1}} < \sigma_{v_{p,\eta 2}}$. Therefore, from $(A.2.12)$ and $(A.2.14)$, we can see that:

$$\begin{aligned} \Pr\left(\hat{\alpha}_{p,\eta^*} + v_{p,\eta 1} \geq 0 | \hat{\alpha}_{p,\eta^*} \geq 0\right) &> \Pr\left(\hat{\alpha}_{p,\eta^*} + v_{p,\eta 2} \geq 0 | \hat{\alpha}_{p,\eta^*} \geq 0\right), \\ \Pr\left(\hat{\alpha}_{p,\eta^*} + v_{p,\eta 1} < 0 | \hat{\alpha}_{p,\eta^*} < 0\right) &> \Pr\left(\hat{\alpha}_{p,\eta^*} + v_{p,\eta 2} < 0 | \hat{\alpha}_{p,\eta^*} < 0\right), \\ -\Pr\left(\hat{\alpha}_{p,\eta^*} + v_{p,\eta 1} < 0 | \hat{\alpha}_{p,\eta^*} \geq 0\right) &> -\Pr\left(\hat{\alpha}_{p,\eta^*} + v_{p,\eta 2} < 0 | \hat{\alpha}_{p,\eta^*} \geq 0\right), \\ -\Pr\left(\hat{\alpha}_{p,\eta^*} + v_{p,\eta 1} \geq 0 | \hat{\alpha}_{p,\eta^*} < 0\right) &> -\Pr\left(\hat{\alpha}_{p,\eta^*} + v_{p,\eta 2} \geq 0 | \hat{\alpha}_{p,\eta^*} < 0\right) \end{aligned} \qquad (A.2.15)$$

Substituting $(A.2.15)$ into $(A.2.11)$ gives:

$$E\left(Q_{p,\eta 1} | Q_{p,\eta^*} = 1\right) - E\left(Q_{p,\eta 1} | Q_{p,\eta^*} = -1\right) > E\left(Q_{p,\eta 2} | Q_{p,\eta^*} = 1\right) - E\left(Q_{p,\eta 2} | Q_{p,\eta^*} = -1\right) \qquad (A.2.16)$$

Finally, substituting this into $(A.2.10)$ and using the definition of B_η from $(A.2.6)$, we get $B_{\eta 1} > B_{\eta 2}$. Q.E.D.

Appendix 3:

This appendix derives a recursive relation for cost function in (11) starting from Eq. (8).

From (8), the posterior mean of investors' estimate of gross skill ϕ_p^K is:

$$\phi_{p,\eta,t}^K = \frac{\nu\,\phi_0 + t\vartheta_{\hat{\alpha},\eta}\,\bar{X}_{p,\eta,t}}{\nu + t\vartheta_{\hat{\alpha},\eta}}, \qquad (A.3.1)$$

where $X_{p,\eta,t} = \hat{\alpha}_{p,\eta,t} + c\big(q_{p,t-1}\big)$.

Under the competitive equilibrium of Berk, Green (2004) we have:

$$c\big(q_{p,t}\big) = \phi_{p,\eta,t}^K \qquad (A.3.2)$$

Rewriting $t\bar{X}_{p,\eta,t}$ as $(t-1)\bar{X}_{p,\eta,t-1} + X_{p,\eta,t}$ which can further be written as $(t-1)\bar{X}_{p,\eta,t-1} + \hat{\alpha}_{p,\eta,t} + c\big(q_{p,t-1}\big)$, Eq. (A.3.1) becomes:

$$\phi_{p,\eta,t}^K = \frac{\nu\,\phi_0 + (t-1)\vartheta_{\hat{\alpha},\eta}\,\bar{X}_{p,\eta,t-1} + \vartheta_{\hat{\alpha},\eta}c\big(q_{p,t-1}\big)}{\nu + t\vartheta_{\hat{\alpha},\eta}} + \frac{\vartheta_{\hat{\alpha},\eta}}{\nu + t\vartheta_{\hat{\alpha},\eta}}\hat{\alpha}_{p,\eta,t}\,. \qquad (A.3.3)$$

From (A.3.2), the competitive equilibrium condition for period $t-1$ will be $c\big(q_{p,t-1}\big) = \phi_{p,\eta,t-1}^K$, where $\phi_{p,\eta,t-1}^K = \frac{\nu\,\phi_0 + (t-1)\vartheta_{\hat{\alpha},\eta}\,\bar{X}_{p,\eta,t-1}}{\nu + (t-1)\vartheta_{\hat{\alpha},\eta}}$ is given by (A.3.1) for $t-1$. Therefore,

$$\phi_{p,\eta,t}^K = \frac{\big[\nu + (t-1)\vartheta_{\hat{\alpha},\eta}\big]\phi_{p,\eta,t-1}^K + \vartheta_{\hat{\alpha},\eta}\phi_{p,\eta,t-1}^K}{\nu + t\vartheta_{\hat{\alpha},\eta}} + \frac{\vartheta_{\hat{\alpha},\eta}}{\nu + t\vartheta_{\hat{\alpha},\eta}}\hat{\alpha}_{p,\eta,t}$$

Upon simplification, we get:

$$\phi_{p,\eta,t}^K = \phi_{p,\eta,t-1}^K + \frac{\vartheta_{\hat{\alpha},\eta}}{\nu + t\vartheta_{\hat{\alpha},\eta}}\hat{\alpha}_{p,\eta,t}\,. \qquad (A.3.4)$$

We can get the recursive relation for cost function from Eq. (A.3.4) by using the competitive equilibrium condition for t and $t-1$. Therefore,

$$c\big(q_{p,t}\big) = c\big(q_{p,t-1}\big) + \frac{\vartheta_{\hat{\alpha},\eta}}{\nu + t\vartheta_{\hat{\alpha},\eta}}\hat{\alpha}_{p,\eta,t}\,.$$

Q.E.D.

Appendix 4:

This appendix derives the components of measurement error in $\hat{\alpha}_{p,\eta,t}$ presented in Eq. (25).

Suppose all returns are in excess of risk-free rate. Net returns of a fund are given as follows:

$$r_{p,t} = \phi_p - c(q_{t-1}) + \sum_{k=1}^{K} \beta_{p,k} \times E\left(F_{k,t}\right) + \sum_{k=1}^{J} \beta_{p,k} \times f_{k,t} + \xi_{p,t}, \qquad (A.4.1)$$

where $f_{k,t} = F_{k,t} - E\left(F_{k,t}\right)$ and under the Berk, Green (2004) equilibrium condition $E_{t-1}\left(\phi_p - c(q_{t-1})\right) = 0$.

η −factor model estimate of $\hat{\alpha}$ is computed as:

$$\hat{\alpha}_{p,\eta,t} = r_{p,t} - \sum_{k=1}^{\eta} \hat{\beta}_{p,k} \times F_{k,t}, \qquad (A.4.2)$$

where $\hat{\beta}$s are estimated from a time series regression with returns data from $t - T$ to $t - 1$.

We consider the cases where the J −factor model nests the K − and η −factor models.

We can decompose $\hat{\alpha}_\eta$ in Eq. (A.4.2) as:

$$\hat{\alpha}_\eta = \alpha^K + \left(\alpha_\eta - \alpha^K\right) + \left(\hat{\alpha}_\eta - \alpha_\eta\right), \qquad (A.4.3)$$

where α^K is the true net alpha which equals $\phi_p - c(q_{t-1})$, α^η is the alpha from an η-factor model without any estimation errors and sampling errors, θ is the model misspecification error and ϵ is the statistical estimation error.

First, we rearrange terms in (A.4.2) to separate out the β measurement error part as:

$$\hat{\alpha}_{p,\eta,t} = r_{p,t} - \sum_{k=1}^{\eta} \beta_{p,k} \times F_{k,t} - \sum_{k=1}^{\eta} (\hat{\beta}_{p,k} - \beta_{p,k}) \times F_{k,t}, \qquad (A.4.4)$$

Substituting the expression for $r_{p,t}$ from (A.4.1) in Eq. (A.4.4) above, we get:

$$\hat{\alpha}_{p,\eta,t} = \left(\phi_p - c(q_{t-1})\right) + \sum_{k=1}^{K} \beta_{p,k} \times E\left(F_{k,t}\right) + \sum_{k=1}^{J} \beta_{p,k} \times f_{k,t} - \sum_{k=1}^{\eta} \beta_{p,k} \times F_{k,t}$$
$$- \sum_{k=1}^{\eta} (\hat{\beta}_{p,k} - \beta_{p,k}) \times F_{k,t} + \xi_{p,t}. \qquad (A.4.5)$$

To identify the misspecification error and statistical estimation error components of $\hat{\alpha}_{p,\eta,t}$, first consider the case where $\eta > K$. Without loss of generality, let the first K factors of the η −factor model be the same as the K −factor model and let the first η factors of the J −factor model be same as the η −factor model. With this, we can simplify (A.4.5) as:

$$\hat{\alpha}_{p,\eta,t} = \left(\phi_p - c(q_{t-1})\right) - \sum_{k=K+1}^{\eta} \beta_{p,k} \times E\left(F_{k,t}\right) + \sum_{k=\eta+1}^{J} \beta_{p,k} \times f_{k,t}$$
$$- \sum_{k=1}^{\eta} (\hat{\beta}_{p,k} - \beta_{p,k}) \times F_{k,t} + \xi_{p,t}. \qquad (A.4.6)$$

The term $-\sum_{k=K+1}^{\eta} \beta_{p,k} \times E\left(F_{k,t}\right)$ is the model misspecification part θ which arises due to using more factors than there are in the true asset pricing model (i.e. due to $\eta > K$). The last three terms constitute the estimation error part ϵ which arise due to (i) omitted common factors, (ii) estimation errors in the betas of included factors and (iii) fund's idiosyncratic returns.

Now consider the case where $\eta < K$. Without loss of generality, let the first η factors of the K-factor asset pricing model be the same as the η-factor model and let the first η factors of the J-factor model be same as the η-factor model. With this, we can simplify (A.4.5) as:

$$\hat{\alpha}_{p,\eta,t} = \left(\phi_p - c(q_{t-1})\right) + \sum_{k=\eta+1}^{K} \beta_{p,k} \times E\left(F_{k,t}\right) + \sum_{k=\eta+1}^{J} \beta_{p,k} \times f_{k,t}$$
$$- \sum_{k=1}^{\eta} (\hat{\beta}_{p,k} - \beta_{p,k}) \times F_{k,t} + \xi_{p,t}. \qquad (A.4.7)$$

The first term in (A.4.7), $\sum_{k=\eta+1}^{K} \beta_{p,k} \times E\left(F_{k,t}\right)$, is the model misspecification part θ which arises due to omitting some priced factors in the estimation model. The last three terms constitute the estimation error part ϵ which arise due to (i) omitted common factors, (ii) estimation errors in the betas of included factors and (iii) fund's idiosyncratic returns.

Putting the two cases together, we have:

$$\theta = \begin{cases} -\sum_{k=K+1}^{\eta} \beta_{k,p} E(F_k) \text{ for } \eta \geq K, \\ \sum_{k=\eta+1}^{K} \beta_{k,p} E(F_k) \text{ for } \eta < K \end{cases},$$

$$\epsilon = \sum_{k=\eta+1}^{J} \beta_{k,p}\, f_{k,t} + \sum_{k=1}^{\eta} \left(\hat{\beta}_{k,p,t} - \beta_{k,p}\right) F_{k,t} + \xi_{p,t} \qquad (A.4.8)$$

which validates the result presented in Eq. (25). Q.E.D.

Appendix 5:

This appendix proves the proposition that $Cov\left(\hat{\alpha}_{p,\eta^*,t}, \Gamma_{p,t}\right) > 0$ when $q_t \leq Q^{max}$.

From Eq. (17), when $q_t \leq Q^{max}$ we have $\frac{d\Gamma_{p,t}}{d\hat{\alpha}_{p,\eta^*,t}} > 0$. Therefore Γ_p is an increasing function of $\hat{\alpha}_{p,\eta^*,t}$ when $q_t \leq Q^{max}$. The above proposition will be true if we show that $Cov\left(X, f(X)\right) > 0$ when $f(X)$ is an increasing function of X.

First, note that covariance between two random variables X, Y can be written as:

$$Cov(X,Y) = E\left[\left(X - E(X)\right)(Y - k)\right], \quad (A.5.1)$$

for any constant k.

Denoting $E(X) = \mu$ and setting $k = f(\mu)$ in (A.5.1), we get:

$$Cov\left(X, f(X)\right) = E[(X - \mu)(f(X) - f(\mu))]. \quad (A.5.2)$$

Since $f(.)$ is an increasing function, by definition $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$.

Therefore, for each draw x_i of X,

$$x_i > \mu \Rightarrow f(x_i) > f(\mu),$$

$$x_i < \mu \Rightarrow f(x_i) < f(\mu)$$

Therefore,

$$(x_i - \mu)\left(f(x_i) - f(\mu)\right) \geq 0 \;\forall\; x_i. \quad (A.5.3)$$

Combining the results in (A.5.2) and (A.5.3), we can conclude that $Cov\left(X, f(X)\right) > 0$ when $f(.)$ is an increasing function.

This result implies $Cov\left(\hat{\alpha}_{p,\eta^*,t}, \Gamma_{p,t}\right) > 0$ when $q_t \leq Q^{max}$. Q.E.D.