Procurement with Cost and Non-Cost Attributes: Cost-Sharing Mechanisms

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Problem Definition. A buyer wants to design a procurement mechanism for awarding a project to one among a set of contractors, each of whom is privately informed about his cost and his estimate of an a priori uncertain non-cost attribute (for instance, the completion time). After the project is completed, the winning contractor's non-cost attribute is realized; the buyer incurs a disutility cost that depends on this realization.
Academic/Practical Relevance. This procurement problem arises in situations such as highway construction projects, where completion times are a major concern, and energy-saving programs, where actual cost savings can only be observed after their execution.
Methodology. The buyer's problem is a two-dimensional mechanism-design problem, for which we seek a solution that is easy to implement in practice and delivers a near-optimal procurement cost.
Results. We propose the class of *cost-sharing mechanisms*, in which the winning contractor is selected based on a single bid in a second-price auction and reimburses a prescribed fraction – referred to as the *cost-sharing fraction* – of the buyer's disutility cost upon completion of the project. We show, both theoretically and numerically, that the best cost-sharing mechanism is near-optimal. Further, we show that its performance remains attractive even when the contractors' cost and non-cost attributes are endogenously determined through their private actions. We also quantify the price of anarchy of the optimal cost-sharing mechanism for uniform and normal beliefs.
Managerial Implications. Our results shed light on how the cost-sharing fraction captures the essential trade-off that the informationally-disadvantaged buyer faces between allocative inefficiency and information rent. We offer intuitive recommendations on the choice of this fraction: The buyer should choose a higher cost-sharing fraction as the information asymmetry on the cost dimension becomes more pronounced than that on the non-cost attribute, or as the number of contractors increases.

Key words: procurement; non-cost attribute, multi-dimensional mechanism design; cost sharing

1. Introduction

In many real-world situations where a firm outsources a project, non-cost attributes such as completion time and quality are important considerations for the firm, in addition to cost. Typically, in such settings, the non-cost attributes are observed by the firm after the project is completed. Motivated by such a context, we consider a buyer who wishes to award a project to one among a set of contractors, each of whom is privately informed about his cost and his estimate of an a priori uncertain non-cost attribute. The uncertainty in the non-cost attribute is resolved after the project is completed and inflicts a disutility (or inconvenience) cost on the buyer. The buyer's problem is that of designing a procurement mechanism that minimizes her expected *total cost*.

Motivating Examples

The highway construction industry offers an excellent example of the procurement problem described above. With a shift in focus from building new roads to 3R work (resurfacing, restoration, and rehabilitation) on existing ones, many projects require frequent road closures and traffic diversions, resulting in a major inconvenience to road users. As Lewis and Bajari (2011) point out, for a $1 million contract to repair a highway, which takes approximately 15 days to complete and affects around 100,000 motorists on any given day, the inconvenience to road users from repairs is as high as $4.5 million. As such, reducing the completion time of the project, thereby mitigating such social-welfare losses, has become a crucial objective in the awarding of contracts for many state agencies throughout the United States (Gupta et al. 2015). While contractors typically have estimates on their completion times, given their past experience and capabilities, the actual completion time may fluctuate due to a variety of reasons, including uncertain weather conditions and early/late completion of sub-contractors' activities. Here, the chosen contractor's completion time for the project serves as an example of the non-cost attribute; the buyer's (i.e., the government's) disutility cost arises from the inconvenience inflicted on road users until the project is completed.

As another prominent instance of multi-attribute procurement, the past two decades have witnessed a rapid growth in the *Energy Service Company* (ESCO) industry in the United States (Stuart 2016). Public institutions and private enterprises looking to improve their buildings and other infrastructure in environmentally-sustainable, low-energy-consumption ways resort to ESCOs for identifying viable solutions and implementing them. When selecting an ESCO, an important criterion, besides the up-front cost of installing the energy-efficient upgrades, is the actual energy savings (benchmarked with the baseline consumption prior to the project) delivered in the years following the installation. As Goldman et al. (2005) point out, energy-efficiency initiatives often generate significant savings; e.g., on a survey of 63 energy-saving projects, high-efficiency lighting reduced energy consumption to 53% of its baseline level. ESCOs typically have the ability to forecast energy savings according to the technology they employ (Vine 2005). However, idiosyncratic usage of a facility by its occupants often causes the realized savings to deviate from the predicted amount. Thus, in this example, the energy savings generated by the project correspond to the non-cost attribute; a lower level of savings inflicts a corresponding disutility cost on the buyer (i.e., the owner of the facility).

Challenges in Designing Effective Mechanisms

The buyer's problem belongs to the class of two-dimensional mechanism-design problems, which, in general, lack analytical tractability and are notoriously challenging to solve. Moreover, in many practical settings, the buyer faces the additional complication that the winning contractor might be able to *manipulate* the buyer's observation of the non-cost attribute. For instance, in the construction example, the winning contractor has the ability to deliberately delay the completion of the project. While one might think that it is "obvious" that no contractor would deliberately cause such a delay, the following argument highlights this possibility. By the Revelation Principle (Myerson 1981), we know that the search for an optimal mechanism can be restricted to direct revelation mechanisms; that is, mechanisms where the contractors are asked to submit their cost and non-cost attributes to the buyer and in which the contractors find it in their best interest to reveal these attributes truthfully. Within the class of such mechanisms, it is plausible, in the construction example, that an optimal mechanism could penalize the contractor for completing the project well in advance of the estimated completion time. In such a situation, the winning contractor would find it beneficial to cause deliberate delays. Thus, this is an example of a phenomenon we call *one-way manipulation*, to emphasize the idea that the winning contractor can deliberately (and in a costless manner) cause the buyer's observation of the completion time to be *inferior* to (i.e., larger than) the actual completion time he could have accomplished (of course, here the contractor does not have the ability to reduce the completion time in a costless manner)[1]. Similarly, in examples where quality is the non-cost attribute, it is likely that the winning contractor has the ability to deliberately make the buyer's observed quality worse than the best-possible quality he can deliver; but, post the implementation of the winning contractor's project, the contractor might have no ability to achieve a higher quality: this is again an example of one-way manipulation. The model we study in Section 3 explicitly allows such one-way manipulation possibilities.

Our Solution: Cost-Sharing Mechanisms

In light of the above difficulties, rather than obtain an optimal mechanism, our goal in this study is to seek a practical mechanism that is simple to implement and is near-optimal. To this end, we propose the class of *cost-sharing mechanisms*. A cost-sharing mechanism is indexed by a single parameter $\alpha \in [0,1]$, which we refer to as the *cost-sharing fraction*. Under a cost-sharing mechanism with the parameter α, the winning contractor reimburses an α fraction of the disutility cost that the buyer incurs upon the completion of the project. To select a contractor, the buyer conducts a second-price sealed-bid auction, in which each contractor, knowing the cost-sharing fraction α, submits a sealed-bid to the buyer. The contractor with the lowest bid wins and is paid the second-lowest bid from which his share of the disutility cost is deducted upon the completion of the project; other contractors offer no services and receive no payment.

The idea behind a cost-sharing mechanism is simple. It enables a contractor to internalize the disutility cost that his non-cost attribute inflicts on the buyer. Unlike mechanisms that require multi-dimensional bids, a contractor only needs to submit a singular bid that synthetically reflects, to an extent, his privately-known cost and non-cost estimate. As such, cost-sharing mechanisms are intuitive and easy to implement in practice. Indeed, several procurement mechanisms currently used in practice resemble cost-sharing mechanisms. In highway construction, for example, one popular method to award repair projects is the *lane-rental* mechanism (Minnesota Department of Transportation 2017), which selects a contractor through an auction[2] and charges the selected contractor a pre-specified daily lane-rental fee as a fraction (say, 10%) of the actual road-user cost, until the project is completed. In energy-efficiency projects, an ESCO is typically selected through a bidding process and then enters into a *shared-savings contract* with the project owner. Under this contract, the actual energy savings realized upon the project's completion are divided between the owner and the ESCO according to an agreed-upon split percentage; e.g., 85% for the ESCO and 15% for the owner (Hawaii State Energy Office 2017). At a higher level, the commonality among these and cost-sharing mechanisms lies in the observation that a competitive bidding mechanism is used to select a contractor, who is then given a performance contract under which the contractor must internalize a fixed fraction of the externality imposed by his non-cost attribute.

Main Results

(**Theorem 1**) Not only are cost-sharing mechanisms appealing in their implementation simplicity, they also capture the essential economic trade-off faced by the buyer. As is common in many situations plagued with information asymmetry, the informationally-disadvantaged buyer faces a dilemma between awarding the project to the most-efficient contractor (working with whom results in the lowest total cost for the buyer) and reducing her information rent (the winning contractor's markup over his total cost). We show that this trade-off is explicitly captured by the cost-sharing fraction α. In particular, as α increases, while the allocation becomes more efficient, the information rent increases. Consequently, an optimal cost-sharing arrangement is one that strikes a balance between these two countervailing forces.

(**Theorem 2**) Together, allocative inefficiency and information rent increase the buyer's total expected cost above the first-best level (i.e., the cost that can be achieved in the absence of information asymmetry). To quantify this increase, we establish the *price of anarchy* (PoA), i.e., the worst-case ratio of the buyer's cost under the optimal cost-sharing mechanism to the first-best cost, under uniform and normal beliefs. Specifically, we show that the PoA under the uniform (resp., normal) belief is 2 (resp., 1.5). These values are reminiscent of similar results for the PoA in the literature on supply-chain contracts and auctions (see e.g., Perakis and Roels 2007, Roughgarden et al. 2017).

(**Theorem 3**) We provide guidelines on the choice of the cost-sharing fraction α in practice. To this end, a key result we show is that the optimal cost-sharing fraction depends only on the relative uncertainties the buyer has about the contractors' costs and their non-cost estimates. More specifically, a higher cost-sharing fraction is recommended when the information asymmetry on the cost becomes more pronounced than that on the non-cost estimate. Our numerical studies further reveal that a more-competitive market (i.e., one with a larger number of contractors) also induces the buyer to choose a higher cost-sharing fraction.

(**Theorem 4**) To qualify cost-sharing mechanisms as a viable solution for the buyer's procurement problem in practice, we demonstrate both theoretically and numerically that the buyer's expected cost under the optimal cost-sharing mechanism is impressively close to that under an optimal mechanism. Specifically, in an environment with uniform beliefs and two contractors (i.e., the least competitive market), we prove that the buyer's expected cost under the optimal cost-sharing mechanism is at most 8.15% more than the optimal cost over all mechanisms. Further, we numerically demonstrate that the optimal cost-sharing mechanism delivers an equally attractive performance for normal beliefs, and that its performance improves as the supply market becomes increasingly competitive.

(**Theorem 5**) For procurement settings where the contractors' cost and non-cost attributes are *endogenously* determined by their costly and private actions, we demonstrate that, as before, the cost-sharing mechanism still acts to balance the rent-efficiency trade-off. More importantly, we again show that the best cost-sharing mechanism is near-optimal, and that the guidelines for choosing the cost-sharing fraction are qualitatively consistent with those recommended for exogenous cost and non-cost attributes.

Organization of the Paper

The plan for the rest of the paper is as follows. We discuss the relevant literature in the next section. The procurement problem we study is formally defined in Section 3, where we also establish lower bounds on the optimal cost for our subsequent analysis. Section 4 is dedicated to the theoretical analysis of cost-sharing mechanisms. This is then supplemented by extensive numerical studies in Section 4.2.1. In Section 5, we allow the contractors' cost and non-cost attributes to be endogenous and verify the robustness of our results. Section 6 concludes. Appendices A and B present auxiliary results and the proofs of the results in Section 4, respectively. The remaining proofs are relegated to the Electronic Companion.

2. Literature Review

Our paper contributes to the well-established field of mechanism design that dates back to the seminal work by Mussa and Rosen (1978) and Myerson (1981). They modeled information asymmetry with a one-dimensional parameter and developed solution procedures that have since become standard in the literature for identifying optimal mechanisms. However, extending this analysis to multi-dimensional private information has proven to be difficult, in general. Here, the main technical challenge arises from resolving the complexity of the binding "non-local" incentive-compatibility constraints (Rochet and Choné 1998, Manelli and Vincent 2007, Belloni et al. 2010).

Instead of identifying an optimal solution for a multi-dimensional mechanism-design problem, researchers typically endeavor to obtain implementable mechanisms that perform well relative to an appropriate benchmark. Many of these mechanisms are variations of the standard price-only auctions (e.g., Krishna 2009) – a prominent example is the scoring-auction (Asker and Cantillon 2008), which specifies a scoring rule to convert a contractor's multi-dimensional bid to a scalar bid for evaluation. As a special type of scoring auctions, A+B auctions (Gupta et al. 2015, Tang et al. 2015) are particularly popular in awarding infrastructural projects, whereby contractors submit two-dimensional bids that are evaluated via a pre-determined scoring rule. On the contrary, beauty-contest auctions (Klemperer 2002) are non-score-based auctions, whereby the buyer solicits multi-dimensional bids on attributes she cares about but announces no specific allocation rule. The cost-sharing mechanism examined in our paper differs from these mechanisms in that the contractors submit one-dimensional bids which "inherently" score their multi-dimensional attributes.

The notion of a cost-sharing mechanism first appeared in Samuelson (1987), which was a comment in response to Hansen (1985). The cost-sharing mechanism also shares some similarities with the total-cost procurement auction studied in Kostamis et al. (2009); however, in their setting, the non-price dimension is known to the buyer whereas, in our problem, it is unknown to the buyer and is subject to random noise.

Two papers – Chen et al. (2010) and Chaturvedi and Martínez-de-Albéniz (2011) – are closely related to our problem. They consider a buyer who procures a product from suppliers differentiated by their costs and reliabilities. A supplier's ability to deliver the product is modelled as a Bernoulli random variable, which takes the value 1 if the product is delivered, and 0 otherwise; the expected value of this random variable is the supplier's reliability. Each supplier is privately informed of his cost and reliability. This setting resonates well with our procurement problem in the sense that the product's delivery can be viewed as the non-cost attribute and a supplier's reliability as his estimate of this attribute. The goal in Chen et al. (2010) is to evaluate scoring auctions with quasi-linear scoring rules in which suppliers bid their costs and penalties for non-delivery. Chaturvedi and Martínez-de-Albéniz (2011), on the other hand, aim to find an optimal sealed-bid mechanism that requires suppliers to report their costs and reliabilities truthfully, and allows a payment function that depends on the suppliers' bids and on whether or not the supplier eventually delivers the product (i.e., contingent upon delivery). An interesting observation in Chaturvedi and Martínez-de-Albéniz (2011) – driven by the fact that a delivery failure can be verified ex-post by the buyer – is that, under a specific set of technical conditions (see Lemma 3 of that paper), suppliers' private information on their reliability is inconsequential and generates no information rent; there exist similar observations in settings with single-dimensional private information and an ex-post verifiable signal; see, e.g., Riordan and Sappington (1998), Crémer and McLean (1988). Our goal is distinct from these two papers in the sense that the cost-sharing mechanism only solicits a *single-dimensional* bid from contractors, who themselves take into account their two-dimensional private information ex-ante while being aware of the contingent payments ex-post.

In the settings that motivate our study, the value of the contract is ex-ante unknown but is ex-post observable to the buyer, naturally leading one to consider mechanisms with contingent payments; see e.g., Skrzypacz (2013) for a survey on commonly-used auctions with contingent payments. This stream of literature typically considers a setting in which the buyer specifies a parameterized family of contracts and the sellers bid on the parameters. In contrast, in our setting, the buyer specifies the cost-sharing fraction upfront.

3. Model

Consider a buyer (firm) who wishes to award a project to one among N contractors, indexed by $n = 1, 2, \ldots, N$. For each n, contractor n is endowed with a two-dimensional private type, (x_n, y_n), where x_n and y_n are realizations of two independent, continuous random variables X_n and Y_n with density (distribution) functions f_X (F_X) and f_Y (F_Y), respectively. Let $\mathbf{X} = (X_1, X_2, \ldots, X_N)$ and $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_N)$. We assume that $f_X(\cdot)$ is log-concave[3] with support $[\underline{x}, \overline{x}]$. The types of the contractors are mutually independent. Here, x_n denotes the cost incurred by contractor n for executing the project and y_n denotes contractor n's *estimate* of an a priori uncertain non-cost attribute, such as completion time. If contractor n is awarded the project, then that contractor will observe the *realization* of this non-cost attribute, denoted by $y_n + \epsilon$, where ϵ is the realization of a random noise with mean zero. Moreover, the contractor has the ability to manipulate this attribute upwards in a costless manner (such as deliberately delaying the completion of the project) so that the buyer *experiences* the value of the non-cost attribute as being

$$z_n \geq y_n + \epsilon$$

and incurs an associated disutility cost of $V(z_n)$, where V is a non-negative and increasing function. The buyer's problem is to design a *feasible mechanism* which minimizes her expected *total cost*; that is, the sum of the payments made to the contractors plus the disutility cost. A mechanism is said to be feasible if (a) with probability 1, the project is awarded to exactly one contractor, and (b) it is *individually rational*; that is, for every (n, x_n, y_n), the equilibrium expected net-profit to contractor n is non-negative. Let us denote this mechanism-design problem by $\mathcal{P}$ and its optimal value by OPT; i.e., the minimum expected total cost to the buyer, over all feasible mechanisms.

Note: While $\mathcal{P}$ can be mathematically formulated using the generalized Revelation Principle (Myerson 1982), the necessary notation is quite burdensome and unnecessary as far as our paper is concerned.

As discussed in Section 1, two-dimensional mechanism-design problems are well-known to be challenging. Moreover, our problem includes the features that the buyer observes the non-cost attribute ex-post (i.e., after the winning contractor has executed the project) and that this observation suffers from the possibility of one-way manipulation by the winning contractor. Rather than pursue an optimal mechanism, we derive simple but cost-effective mechanisms (in Section 4). To gain some intuition to help us derive such mechanisms, we first study (in Sections 3.1–3.3) some special settings/mechanism-design problems related to $\mathcal{P}$. In this process, we will also obtain a useful lower bound on OPT (in Section 3.2). Section 3.4 summarizes our findings in Sections 3.1–3.3.

3.1. First-Best Cost

The *first-best* setting is one in which there is no information asymmetry among the parties, i.e., every contractor's cost and non-cost estimate are known to the buyer. Let $\mathbf{x}$ and $\mathbf{y}$ denote, respectively, the cost-attribute vector and the non-cost estimate vector. In this setting, it suffices for the buyer to compensate only the contractor who is awarded the project. Moreover, the winning contractor needs to be compensated only for his cost; that is, there is no need for the buyer to leave a strictly positive profit or *information rent*, to the winning contractor. In other words, this setting results in a *centralized problem*; here, all the contractors are resources belonging to the buyer, who then picks the most efficient resource to execute the project; that is, $\arg\min_n \{x_n + E_\epsilon[V(y_n+\epsilon)]\}$. Here, the minimand is the *centralized cost* when contractor n executes the project. For brevity, let

$$FB(\mathbf{x},\mathbf{y}) = \min_n \{x_n + E_\epsilon[V(y_n+\epsilon)]\}$$

denote the minimum centralized cost or *first-best cost* as a function of $(\mathbf{x},\mathbf{y})$ and let

$$FB = E_{(\mathbf{X},\mathbf{Y})}\left[FB(\mathbf{X},\mathbf{Y})\right]$$

denote the *expected first-best cost*. Consider any feasible mechanism in $\mathcal{P}$. For every realization of $(\mathbf{X},\mathbf{Y})$, the total cost incurred by the buyer under this mechanism includes the centralized cost plus the information rents or profits paid to the contractors. Therefore, the first-best cost is a lower bound on OPT, the optimal expected cost in $\mathcal{P}$; that is,

$$FB \le OPT. \tag{1}$$

3.2. Optimal Cost when y is Public Information

Next, consider the case where the non-cost estimate vector, $\mathbf{y}$, is public information *and* the buyer's observation of the winner's non-cost attribute is *non-manipulable*. Thus, ex-ante, the buyer knows the expected disutility cost, $w_n := E_\epsilon[V(y_n+\epsilon)]$, incurred if she awards the project to contractor n; ex-post, the buyer observes ϵ. Thus, the buyer now faces a mechanism-design problem with single-dimensional private types. Let $\mathbf{W}$ denote the random vector $(W_1, W_2, \ldots, W_N)$ where, for any n, $W_n := E_\epsilon[V(Y_n+\epsilon)]$. Also, let $\mathbf{w}$ denote the vector $(w_1, w_2, \ldots, w_N)$ corresponding to a given realization $\mathbf{y}$ of $\mathbf{Y}$. Let $\mathcal{P}_{NC}(\mathbf{w})$ denote the corresponding mechanism-design problem; the subscript "NC" stands for "non-cost" and signifies that $\mathbf{w}$ is public information. Let $OPT_{NC}(\mathbf{w})$ denote the optimal cost in $\mathcal{P}_{NC}(\mathbf{w})$ and let $OPT_{NC} = E_{\mathbf{W}}[OPT_{NC}(\mathbf{W})]$ denote the expected optimal cost.

The following mechanism is optimal in this setting (Kostamis et al. 2009): Contractors are required to report their costs. Let $\hat{x}_n$ denote contractor n's report of his true cost attribute, x_n. The project is awarded to

$$\arg\min_n \{\psi(\hat{x}_n) + w_n\},$$

where $\psi(x) := x + F_X(x)/f_X(x)$ is the virtual-cost function (Myerson 1981). Only the winning contractor is paid and he is paid an amount equal to the highest cost he could have reported and still won the project. That is, if contractor n wins the project, he is paid

$$\sup\left\{x \in [\underline{x}, \overline{x}]:\ \psi(x) + w_n \leq \psi(\hat{x}_{n'}) + w_{n'} \ \ \forall\ n' \neq n\right\}.$$

Using standard arguments in mechanism design, the following two conclusions can be drawn: (i) This mechanism is *incentive compatible*; that is, it is a Bayesian-Nash equilibrium for every contractor to report his cost attribute truthfully (in fact, it is a dominant strategy to do so). (ii) For every $\mathbf{w}$, the expected cost is $E_{\mathbf{X}}[\min_n\{\psi(X_n) + w_n\}]$. Since this mechanism is optimal for problem $\mathcal{P}_{NC}(\mathbf{w})$ (Kostamis et al. 2009), we have

$$OPT_{NC}(\mathbf{w}) = E_{\mathbf{X}}[\min_n\{\psi(X_n) + w_n\}], \tag{2}$$

and, therefore,

$$OPT_{NC} = E_{(\mathbf{X},\mathbf{W})}\left[\min_n\{\psi(X_n) + W_n\}\right]. \tag{3}$$

It should be clear that OPT_{NC} is a lower bound on OPT – an intuitive explanation follows. Recall that $OPT_{NC}(\mathbf{w})$ and OPT are the optimal costs for mechanism-design problems $\mathcal{P}_{NC}(\mathbf{w})$ and $\mathcal{P}$, respectively. In the former, $\mathbf{y}$ (and hence $\mathbf{w}$) is public information and the buyer's observation of the winner's non-cost attribute is non-manipulable whereas, in the latter, $\mathbf{y}$ is private information of the contractors and the possibility of manipulation exists; hence, the claim. Formally,

$$OPT_{NC} \leq OPT. \tag{4}$$

A rigorous proof of this assertion, using the Revelation Principle, is straightforward but cumbersome and is therefore omitted. Similarly, it is also easy to see that

$$FB \leq OPT_{NC}. \tag{5}$$

Thus, OPT_{NC} is a tighter lower bound (than FB) on OPT. Since OPT is difficult to express analytically, we will use OPT_{NC} to measure the cost effectiveness of *cost-sharing* mechanisms in Section 4.

We now derive a new mechanism in which the winning contractor is required to bear a 100% share of the buyer's disutility cost, and show that it is also optimal for $\mathcal{P}_{NC}(\mathbf{w})$. In this mechanism, only the winning contractor is paid, and, that contractor, say n, bears the *entire responsibility* for the disutility cost $V(y_n + \epsilon)$ realized after he completes the project. Keeping this in mind, the contractors are asked to report their expected costs of executing the project. Since the true expected cost of executing the project to contractor n is

$$T_n = X_n + w_n,$$

the contractors bids are essentially reports of the realizations $(t_1, t_2, \ldots, t_N)$ of the "modified types" $(T_1, T_2, \ldots, T_N)$. Let $\hat{t}_n$ denote contractor n's report of t_n. Since y_n (and hence w_n) is publicly known, the distribution of T_n is the same as the distribution of X_n with a shift of w_n. Thus, if ψ_n denotes the virtual-cost function for T_n, then,

$$\psi_n(t) = \psi(t - w_n) + w_n \quad \forall t \in [\underline{x} + w_n, \overline{x} + w_n].$$

The project is awarded to the lowest virtual-cost bidder, that is, $\arg\min_n \psi_n(\hat{t}_n)$. If contractor n wins the project, then he is paid the highest value of $\hat{t}_n$ at which he would have won; that is,

$$\sup\left\{t \in [\underline{x} + w_n, \overline{x} + w_n] : \ \psi_n(t) \le \psi_{n'}(\hat{t}_{n'}) \quad \forall n' \neq n\right\}.$$

After the project is executed, the winning contractor reimburses the buyer her disutility cost $V(y_n + \epsilon)$. Similar to the mechanism described earlier in this subsection, it is easy to use standard arguments in mechanism design to obtain the following conclusions: (i) The current mechanism is also incentive compatible – that is, it is a Bayesian-Nash Equilibrium for every contractor n to report his type t_n truthfully. (ii) For every realization of $\mathbf{w}$, the expected cost incurred by the buyer under this mechanism is $E_{\mathbf{X}}[\min_n\{\psi_n(X_n + w_n)\}]$, which is easily seen to be the same as $E_{\mathbf{X}}[\min_n\{\psi(X_n) + w_n\}]$ and hence equals $OPT_{NC}(\mathbf{w})$ from (2). Thus, this mechanism is also optimal for $\mathcal{P}_{NC}(\mathbf{w})$.

3.3. Optimal Cost when x is Public Information

Now, consider the case where the realization, $\mathbf{x}$, of the cost attribute vector, $\mathbf{X}$, is public information. The estimates $\mathbf{Y}$ of the non-cost attribute remain private to the contractors, with possible ex-post manipulation of this attribute by the selected contractor. Given $\mathbf{x}$, the resulting mechanism-design problem for the buyer is denoted by $\mathcal{P}_C(\mathbf{x})$ (the subscript "C" stands for "cost" and signifies that $\mathbf{x}$ is public information). Let $OPT_C(\mathbf{x})$ denote the optimal cost in $\mathcal{P}_C(\mathbf{x})$ and let $OPT_C = E_{\mathbf{X}}[OPT_C(\mathbf{X})]$ denote the expected optimal cost. Recalling our discussion of the first-best setting, it can be seen that when $\mathbf{x}$ is public information, $E_{\mathbf{Y}}[FB(\mathbf{x},\mathbf{Y})]$ is a lower bound on $OPT_C(\mathbf{x})$. We will now identify a feasible mechanism that achieves this lower bound for every $\mathbf{x}$, thus proving the optimality of that mechanism for problem $\mathcal{P}_C(\mathbf{x})$.

Consider the following mechanism. The buyer asks each contractor to report his non-cost estimate. Let $\hat{y}_n$ denote contractor n's report. The project is awarded to $\arg\min_n x_n + E_\epsilon[V(\hat{y}_n+\epsilon)]$. The winning contractor, say n, is paid his cost x_n; other contractors are not paid. Here, the winner takes on a 0% share of the buyer's disutility cost. Under this mechanism, it is clear that each contractor makes a profit of zero, regardless of his report. Thus, it is a Bayesian-Nash Equilibrium (in fact, a dominant strategy) for all contractors to report their types truthfully and the selected contractor has no incentive to manipulate the ex-post realization of his non-cost attribute. Thus, for every $(\mathbf{x},\mathbf{y})$, the buyer's total cost is $\min_n x_n + E_\epsilon[V(y_n+\epsilon)] = FB(\mathbf{x},\mathbf{y})$. Therefore, for every $\mathbf{x}$, the buyer's expected total cost is $E_{\mathbf{Y}}[FB(\mathbf{x},\mathbf{Y})]$.

Combining the observations in the two paragraphs above, we see that the mechanism defined above, in which the winning contractor bears a 0% share of the disutility cost, is optimal for $\mathcal{P}_C(\mathbf{x})$. Moreover, $OPT_C(\mathbf{x}) = E_{\mathbf{Y}}[FB(\mathbf{x},\mathbf{Y})]$ for every $\mathbf{x}$; therefore, $OPT_C = FB$.

3.4. Summary

Below, we summarize the main takeaways from our discussion in this section.

- When the non-cost estimate vector is public information, then there is an optimal mechanism in which the winning contractor bears a 100% share of the disutility cost. In contrast, when the cost attribute vector is public information, then we obtained an optimal mechanism in which the contractor bears a 0% share of the disutility cost. Thus, these two mechanisms represent the two extremes of the contractor's share of the disutility cost. When we study cost-sharing mechanisms in Section 4, these observations motivate us to explore the connection between the contractor's "optimal" share of the disutility cost and the extent of information asymmetry in the cost and the non-cost estimate.

- We obtained three lower bounds, namely FB (Section 3.1), OPT_{NC} (Section 3.2), and OPT_C (Section 3.3), on the optimal cost of problem $\mathcal{P}$. Since OPT_{NC} is the tightest of these three lower bounds, we will use it subsequently to establish the near-optimality of cost-sharing mechanisms. We will also make use of the first-best cost to quantify the cost increment due to information rent and allocative inefficiency under a cost-sharing mechanism.

4. Cost-Sharing Mechanisms

Recall from Section 1 that under the cost-sharing mechanism with parameter $\alpha \in [0,1]$, hereafter denoted by CS_α, the selected contractor reimburses an α fraction of the buyer's realized disutility cost. To select a contractor, the buyer conducts a second-price sealed-bid auction, in which every contractor submits a sealed-bid to the buyer and the contractor with the lowest bid wins. The selected contractor is paid the second-lowest bid, from which his share of the disutility cost is deducted upon the project's completion; other contractors receive no payment.

The selected contractor, say n, bears an α fraction of the buyer's disutility cost $V(z_n)$, which is an increasing function of z_n (the value the buyer experiences of the selected contractor's non-cost attribute). Thus, it is immediate that the contractor has no incentive to inflate his realized non-cost attribute; that is, $z_n = y_n + \epsilon$. That is, CS_α is a *manipulation-free* mechanism. Thus, under this mechanism, the total cost incurred by the contractor is $x_n + \alpha \mathbb{E}_\epsilon[V(y_n+\epsilon)] = x_n + \alpha w_n$. Then, standard arguments in second-price sealed-bid auctions lead to the following result.

Lemma 1. *Under CS_α, it is a Bayesian-Nash Equilibrium (in fact, a dominant strategy) for contractor n to bid $x_n + \alpha \mathbb{E}_\epsilon[V(y_n+\epsilon)]$, $n = 1, 2, \ldots N$. Further, CS_α is manipulation-free.*

The next two subsections investigate fundamental properties of cost-sharing mechanisms.

4.1. Allocative Inefficiency *vs.* Information Rent

Let us denote by $C(\alpha)$ the buyer's expected cost under CS_α. We denote the l-th lowest order statistic of $\{X_n + \alpha W_n : n = 1, \cdots, N\}$ by $X_{l(\alpha)} + \alpha W_{l(\alpha)}$; that is, $X_{n(\alpha)} + \alpha W_{n(\alpha)} \leq X_{m(\alpha)} + \alpha W_{m(\alpha)}$ for all $n < m$. The buyer's expected cost is:

$$C(\alpha) = \mathbb{E}\left[X_{2(\alpha)} + \alpha W_{2(\alpha)} + (1-\alpha) W_{1(\alpha)}\right], \tag{6}$$

where the sum of the first two terms within the expectation operator is the payment made to the winning contractor and the third term is the portion of the disutility cost borne by the buyer. Therefore, an optimal mechanism within the family of cost-sharing mechanisms is determined by the fraction α^{opt}, where $\alpha^{opt} := \arg\min_\alpha C(\alpha)$. Note that the first-best expected cost can be expressed as

$$FB = \mathbb{E}\left[X_{1(1)} + W_{1(1)}\right].$$

Thus, the deviation of $C(\alpha)$ from the first-best cost can be written as:

$$C(\alpha) - \mathbb{E}\left[X_{1(1)} + W_{1(1)}\right] = \underbrace{\mathbb{E}\left[X_{1(\alpha)} + W_{1(\alpha)} - X_{1(1)} - W_{1(1)}\right]}_{\text{allocative inefficiency}} + \underbrace{\mathbb{E}\left[X_{2(\alpha)} + \alpha W_{2(\alpha)} - X_{1(\alpha)} - \alpha W_{1(\alpha)}\right]}_{\text{information rent}}.$$

The first term on the right-hand-side represents an incremental cost due to the *allocative inefficiency* in awarding the contract to contractor $1(\alpha)$ instead of the most efficient contractor $1(1)$. The second term represents the surplus given up by the buyer to the selected contractor $1(\alpha)$, whose total cost is $X_{1(\alpha)} + \alpha W_{1(\alpha)}$, but is paid $X_{2(\alpha)} + \alpha W_{2(\alpha)}$. This surplus is the *information rent* paid to the contractor.

The following result explains how the choice of α affects the allocative inefficiency and the information rent.

THEOREM 1. *Under the family of cost-sharing mechanisms $\{CS_\alpha : 0 \le \alpha \le 1\}$, as the cost-sharing fraction α increases, the allocative inefficiency reduces and the information rent increases.*

Under CS_α, we have argued that every contractor bids his *total* cost, which is a realization of the random variable $X_n + \alpha W_n$. As the cost-sharing fraction α goes up, $X_n + \alpha W_n$ approaches $X_n + W_n$. In the extreme case, when $\alpha = 1$, the contractor with the lowest $X_n + W_n$ is selected, thus achieving a fully efficient allocation. On the other hand, a higher cost-sharing fraction α renders the distribution of the contractor's total cost $X_n + \alpha W_n$ more dispersed, increasing the information asymmetry faced by the buyer. As a result, the buyer has to give up higher information rent to the winning contractor.

In essence, Theorem 1 establishes the important role of the cost-sharing fraction α – a single parameter – in striking the classical trade-off between information rent and allocative inefficiency. Despite this simple intuition, the proof of this result is interesting in its own right – it neatly combines ideas from dispersive orderings of random variables and order statistics. We present the proof below.

Proof of Theorem 1

We first prove that the expected allocative inefficiency is decreasing in α. Recall that $W_n = \mathbb{E}_\epsilon[V(Y_n + \epsilon)]$. Since Y_n is identically distributed across all n, so is W_n. We need to show that

$$\mathbb{E}[X_{1(\alpha_1)} + W_{1(\alpha_1)}] \ge \mathbb{E}[X_{1(\alpha_2)} + W_{1(\alpha_2)}] \ \forall\ 0 \le \alpha_1 \le \alpha_2 \le 1. \tag{7}$$

We will show this by proving the following stronger claim: Consider an arbitrary realization of the sequence $(X_n, W_n)_{n\in\mathcal{N}}$, which we denote by $(x_n, w_n)_{n\in\mathcal{N}}$. We claim that

$$x_{1(\alpha_1)} + w_{1(\alpha_1)} \ge x_{1(\alpha_2)} + w_{1(\alpha_2)}. \tag{8}$$

By definition, we know that

$$x_{1(\alpha_1)} + \alpha_1 w_{1(\alpha_1)} \le x_{1(\alpha_2)} + \alpha_1 w_{1(\alpha_2)} \quad \text{and} \quad x_{1(\alpha_2)} + \alpha_2 w_{1(\alpha_2)} \le x_{1(\alpha_1)} + \alpha_2 w_{1(\alpha_1)}.$$

These two statements imply that

$$\alpha_2[w_{1(\alpha_2)} - w_{1(\alpha_1)}] \le x_{1(\alpha_1)} - x_{1(\alpha_2)} \le \alpha_1[w_{1(\alpha_2)} - w_{1(\alpha_1)}].$$

Since $\alpha_2 \ge \alpha_1$, we obtain

$$w_{1(\alpha_2)} - w_{1(\alpha_1)} \le 0.$$

Using these inequalities, we obtain:

$$\begin{aligned}
[x_{1(\alpha_1)} + w_{1(\alpha_1)}] - [x_{1(\alpha_2)} + w_{1(\alpha_2)}] &= [x_{1(\alpha_1)} - x_{1(\alpha_2)}] + [w_{1(\alpha_1)} - w_{1(\alpha_2)}] \\
&\ge \alpha_2[w_{1(\alpha_2)} - w_{1(\alpha_1)}] + [w_{1(\alpha_1)} - w_{1(\alpha_2)}] \\
&= (1-\alpha_2)[w_{1(\alpha_1)} - w_{1(\alpha_2)}] \ge 0.
\end{aligned}$$

This proves the claimed inequality in (8), which in turn implies the desired result (7). This completes our proof for the first part of Theorem 1.

We now prove that the expected information rent under the cost-sharing mechanism is increasing in α. To prove this result, we recall several definitions and results from the stochastic ordering literature (see Section 3.B.2 in Shaked and Shanthikumar 2007 for details):

Definition 1. Let X and Y be two random variables such that $\mathbb{P}(X > x) \le \mathbb{P}(Y > x)$ for all $x \in (-\infty, \infty)$. Then, X is said to be *smaller than Y in the usual stochastic order* (denoted by $X \le_{st} Y$).

Definition 2. Let X and Y be two random variables with distribution functions F and G, respectively. Let F^{-1} and G^{-1} be the right continuous inverses of F and G, respectively. The random variable X is said to be *smaller in dispersive order* than Y, denoted by $X \le_{disp} Y$, if $F^{-1}(b) - F^{-1}(a) \le G^{-1}(b) - G^{-1}(a)$ for all $0 < a \le b < 1$.

Definition 3. A random variable Z is said to be *dispersive* if $X + Z \le_{disp} Y + Z$ whenever $X \le_{disp} Y$, and Z is independent of X and Y.

Proposition 1. *(Shaked and Shanthikumar 2007) If $X \le_{st} Y$, then $\mathbb{E}[X] \le \mathbb{E}[Y]$.*

Proposition 2. *(Shaked and Shanthikumar 2007) Let X be a random variable. Then $X \le_{disp} aX$ for all $a \ge 1$.*

Proposition 3. *(Shaked and Shanthikumar 2007) The random variable X is dispersive if and only if X has a log-concave density.*

PROPOSITION 4. *(Bartoszewicz 1986) Let X and Y be two random variables with distribution functions F and G, respectively. Let $X_{(1)}, X_{(2)}, \ldots, X_{(N)}$ and $Y_{(1)}, Y_{(2)}, \ldots, Y_{(N)}$ denote order statistics of random samples of size N from distributions F and G, respectively. If $X \leq_{disp} Y$, then $X_{(n)} - X_{(n-1)} \leq_{st} Y_{(n)} - Y_{(n-1)}$ for all $n = 1, 2, \ldots, N$, where $X_{(0)} = \inf\{x : F(x) > 0\}$ and $Y_{(0)} = \inf\{y : G(y) > 0\}$.*

We now use the above definitions and results to prove that the expected information rent under the cost-sharing mechanism is increasing in α. Consider an arbitrary α_1, α_2, where $\alpha_2 \geq \alpha_1$. Let

$$I(\alpha) = \mathbb{E}\left[X_{2(\alpha)} + \alpha W_{2(\alpha)} - X_{1(\alpha)} - \alpha W_{1(\alpha)}\right]$$

denote the expected information rent. We are required to show that $I(\alpha_2) \geq I(\alpha_1)$. From Proposition 2, we have $\alpha_1 W_n \leq_{disp} \alpha_2 W_n$ for all n. Given that f_X is log-concave and the assumption that, for all n, X_n and W_n are independent, using Proposition 3, we obtain

$$X_n + \alpha_1 W_n \leq_{disp} X_n + \alpha_2 W_n \quad \forall\ n.$$

This, together with Proposition 4 implies that

$$X_{2(\alpha_1)} + \alpha_1 W_{2(\alpha_1)} - X_{1(\alpha_1)} - \alpha_1 W_{1(\alpha_1)} \leq_{st} X_{2(\alpha_2)} + \alpha_2 W_{2(\alpha_2)} - X_{1(\alpha_2)} - \alpha_2 W_{1(\alpha_2)}.$$

Applying Proposition 1 to this result yields the desired inequality $I(\alpha_1) \leq I(\alpha_2)$. ■

Having understood the trade-off that the cost-sharing fraction aims to balance, we now turn to the characterization of the optimal cost-sharing mechanism. While this is a challenging task in general, we accomplish it for two distributions – namely, uniform and normal – that have been commonly used for modeling informational uncertainty in the contracting literature (see e.g., Davis et al. 2014, Li and Wan 2016, Freyberger and Larsen 2017). For both the classes defined below, we obtain closed-form expressions of $C(\alpha)$ (see (A.1) and (A.4)) and characterize α^{opt} (see (A.2), (A.3) and (A.5)).

Class $\mathcal{P}_U$: Consider the special case in which there are $N = 2$ contractors with X_n and W_n *uniformly* distributed, respectively on $[\underline{x}, \bar{x}]$ and $[\underline{w}, \overline{w}]$ $(n = 1, 2)$. Let $\mu_X = (\underline{x} + \overline{x})/2$ and $\delta_X = \overline{x} - \underline{x}$ be the mean and spread of X_n, respectively. Similarly, let $\mu_W = (\underline{w} + \overline{w})/2$ and $\delta_W = \overline{w} - \underline{w}$. Assume that $\underline{x} \geq 0$ and $\underline{w} \geq 0$ so that the realizations of X_n and W_n are non-negative for any n.

Class $\mathcal{P}_N$: Consider the special case in which there are $N = 2$ contractors with X_n and W_n *normally* distributed $(n = 1, 2)$. Let μ_X and σ_X be the mean and standard deviation of X_n, respectively. Similarly, let μ_W and σ_W be the mean and standard deviation of W_n, respectively. Assume that

$\mu_X \geq 3\sigma_X$ and $\mu_W \geq 3\sigma_W$ so that the realizations of X_n and W_n are most likely non-negative for any n.

Together, allocative inefficiency and information rent increase the buyer's procurement cost above the first-best level $FB = \mathbb{E}\left[X_{1(1)} + W_{1(1)}\right]$. To quantify this increase, the result below uses the well-known notion of the *price of anarchy* (PoA) — the worst-case ratio of the buyer's total expected cost $C(\alpha^{opt})$ under the optimal cost-sharing mechanism to the first-best cost FB — for both the classes $\mathcal{P}_U$ and $\mathcal{P}_N$. For similar results on the PoA in the literature on supply chain contracting and auctions, see, e.g., Perakis and Roels (2007), Martínez-de-Albéniz and Simchi-Levi (2009) and Roughgarden et al. (2017).

THEOREM 2. *The price of anarchy is equal to 2 for $\mathcal{P}_U$ and $(3\sqrt{\pi}+1)/(3\sqrt{\pi}-1)$ for $\mathcal{P}_N$.*

The magnitude of these bounds is comparable to PoA of 4/3 in transportation networks (Roughgarden and Tardos 2002) and PoA of $(1-1/e) \approx 0.63$ for first-price sealed bid auctions (Roughgarden et al. 2017), where, in the latter work, the PoA applies to a maximization problem. The PoA in Theorem 2 is achieved when the disutility costs W_n, $n = 1, 2$ become concentrated at 0. As such, the cost-sharing mechanism is closer to the first best in the realistic setting where the disutility costs are not degenerate.

4.2. Optimal Cost-Sharing Fraction and Effectiveness of Cost-Sharing Mechanisms

In this section, we provide guidelines for setting the cost-sharing fraction in practice, and demonstrate the effectiveness (or near-optimality) of cost-sharing mechanisms.

THEOREM 3. *The optimal cost-sharing fraction α^{opt} only depends on and increases with δ_X/δ_W for class $\mathcal{P}_U$, and σ_X/σ_W for class $\mathcal{P}_N$.*

Note that the optimal cost-sharing fraction α^{opt} is independent of the mean locations of X_n and W_n, and is only a function of their relative variability, which is captured by δ_X/δ_W and σ_X/σ_W under uniform and normal distributions, respectively. Indeed, as shown in Theorem 1, the optimal cost-sharing fraction α^{opt} is determined by trading off information rent $\mathbb{E}\left[X_{2(\alpha)} + \alpha W_{2(\alpha)} - \left(X_{1(\alpha)} + \alpha W_{1(\alpha)}\right)\right]$ with allocative inefficiency $\mathbb{E}\left[X_{1(\alpha)} + W_{1(\alpha)} - \left(X_{1(1)} + W_{1(1)}\right)\right]$, in both of which the effects of mean locations get cancelled. Since this trade-off remains unaffected if we scale both X_n and W_n by a constant factor, the optimal fraction α^{opt} must only depend on the relative variability of X_n and W_n, irrespective of their underlying distributions.

To appreciate the significance of Theorem 3, note that information rent is lowered if either the uncertainty in W_n shrinks with the uncertainty in X_n fixed, or the uncertainty in X_n decreases with the uncertainty in W_n fixed. The former condition increases the relative variability, whereas the latter does the opposite. As such, it is not obvious how the information rent changes with the relative variability between X_n and W_n. By the same token, the effect of their relative variability on the allocative inefficiency is not straightforward either. Theorem 3 successfully compares these two effects: when the cost attribute becomes more uncertain relative to the non-cost attribute, the marginal gain in efficiency from choosing a higher cost-sharing fraction α dominates the marginal increment in the information rent, leading to a larger optimal fraction α^{opt}.

The monotonicity result in Theorem 3 provides a practical rule-of-thumb for buying firms in setting their cost-sharing fraction, depending on the informational environment they reside in: the buyer should let the selected contractor reimburse a higher fraction of her disutility cost, if the information asymmetry on the cost dimension is more pronounced than that on the non-cost dimension. This is the case, for instance, when the buyer has already acquired information on the non-cost attributes (e.g., quality) of the contractors during a prior qualification stage. On the other hand, a smaller cost-sharing fraction is recommended when costs are fairly homogenous across the supply base but the non-cost aspects are highly dependent on contractors' heterogeneous expertise and experience.

We now focus on the performance of the optimal cost-sharing mechanism. For class $\mathcal{P}_U$, where X_n follows a uniform distribution, the linearity of the virtual-cost function corresponding to X_n allows us to obtain a closed-form expression for the lower bound OPT_{NC} on the optimal cost OPT. Using this along with the closed-form expression for $C(\alpha)$ (see (A.1)), we establish the following result.

Theorem 4. *For class $\mathcal{P}_U$, the relative optimality gap under the optimal cost-sharing mechanism is given by*

$$\frac{C(\alpha^{opt}) - OPT}{OPT} \leq 8.15\%.$$

In particular, for $\delta_W \leq 2\delta_X$, this gap improves to 2.92%.

Theorem 4 shows that the optimal cost-sharing mechanism performs extremely well for class $\mathcal{P}_U$. Its performance relative to the lower bound OPT_{NC} is worst when the disutility cost W_n is much more uncertain relative to the cost X_n. This is intuitive since OPT_{NC} represents the expected cost when the disutility costs W_n are publicly known. Further, when the uncertainty in W_n is not too excessive compared to that in X_n (i.e., when $\delta_W \leq 2\delta_X$), the performance of the best cost-sharing mechanism improves significantly from 8.15% to 2.92%.

For class $\mathcal{P}_N$, although we have a closed-form expression for $C(\alpha)$ (see (A.4)), a similar characterization for OPT_{NC} is difficult to obtain due to the nonlinearity of the virtual-cost function, preventing us from analytically establishing the near-optimality result. However, as our numerical results in Section 4.2.1 demonstrate, the optimal cost-sharing mechanism remains equally attractive for normal beliefs.

In our preceding analysis, we worked with $N = 2$ contractors, which admittedly lends analytical tractability. Nevertheless, this setting is quite meaningful in that with such a least-competitive supply market, one would expect a conservative performance guarantee on cost-sharing mechanisms. We now conduct a comprehensive numerical study to demonstrate the robustness of the following two key insights with respect to the size of the supply base: (i) α^{opt} increases as the non-cost estimate Y_n becomes more certain compared to the cost X_n, and (ii) $C(\alpha^{opt})$ is near-optimal. We will also see that as the market becomes more competitive (i.e., as the number of contractors increases), the optimal fraction α^{opt} increases, and the performance of the corresponding cost-sharing mechanism improves.

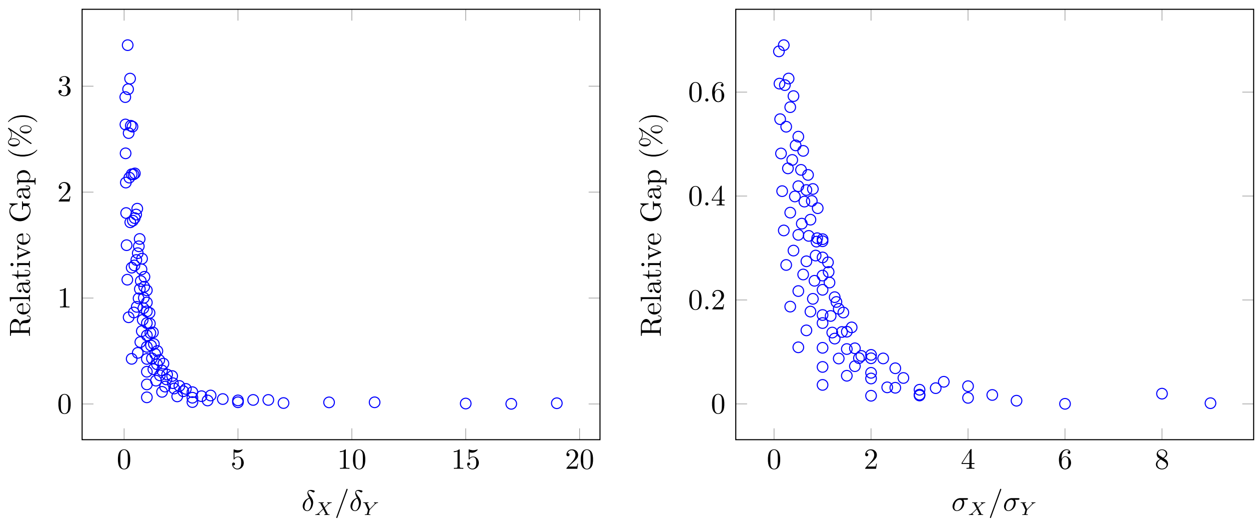
4.2.1. Numerical Study In our test-bed, we assume that the buyer's disutility cost takes a linear functional form $V(\hat{y}) = \hat{y}$, and hence $\mathbf{W} = \mathbf{Y}$. We consider two settings, each of which contains 100 instances. For each instance below, the number of contractors N varies from 2 to 10. In total, we thus have $9 \times (100 + 100) = 1800$ instances in the test-bed.

1. In the first set of instances, we assume that the contractors' costs and non-cost estimates are both uniformly distributed. In particular, X_n follows a uniform distribution with support $[\mu_X - \delta_X/2, \mu_X + \delta_X/2]$, where we fix $\mu_X = 10$ and allow δ_X to take the following ten possible values: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19. Similarly, Y_n follows a uniform distribution with support $[\mu_Y - \delta_Y/2, \mu_Y + \delta_Y/2]$, where we fix $\mu_Y = 10$ and allow δ_Y (which equals δ_W, since V is linear) to take the following ten possible values: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19. This results in $10 \times 10 = 100$ instances.

2. In the second set of instances, we assume that the contractors' costs and non-cost estimates are both normally distributed. In particular, X_n follows a normal distribution, where we fix its mean $\mu_X = 10$ and allow its standard deviation σ_X to take the following ten possible values: 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4, 1.6, 1.8, 2.0. Similarly, Y_n follows a normal distribution, where we fix its mean $\mu_Y = 10$ and allow its standard deviation σ_Y to take the following ten possible values: 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4, 1.6, 1.8, 2.0. This results in $10 \times 10 = 100$ instances.

In the test-bed above, we do not vary μ_X and μ_Y because, as shown in Theorem 3, the optimal cost-sharing parameter α^{opt} does not depend on the means μ_X and μ_Y.

Performance of Cost-Sharing Mechanisms. For every instance in our test-bed, we numerically compute the percentage increment in the cost from using the best cost-sharing mechanism with respect to the lower bound OPT_{NC}, i.e., $100[C(\alpha^{opt}) - OPT_{NC}]/OPT_{NC}$, a small value of which implies an even smaller gap with respect to the optimal cost OPT. Figure 1 plots this gap against δ_X/δ_Y for uniform distributions and against σ_X/σ_Y for normal distributions, for the instances corresponding to $N = 2$ contractors. In line with our discussion in Section 4, we observe that the gap increases when the non-cost estimate becomes more uncertain relative to the cost attribute, for both sets of distributions.



(a) $X_n \sim$ Uniform and $Y_n \sim$ Uniform

(b) $X_n \sim$ Normal and $Y_n \sim$ Normal

Figure 1 **A graphical illustration of the relative gap $100[C(\alpha^{opt}) - OPT_{NC}]/OPT_{NC}$ for each instance of (δ_X, δ_Y) for uniform distributions and (σ_X, σ_Y) for normal distributions, when there are $N = 2$ contractors.**

For each size of the supply base N, Table 1 reports the average and the maximum value of the percentage gap $100[C(\alpha^{opt}) - OPT_{NC}]/OPT_{NC}$ over the 100 instances of (δ_X, δ_Y) for uniform distributions, and the 100 instances of (σ_X, σ_Y) for normal distributions, respectively. When the cost and non-cost estimates are uniformly distributed, the gap takes an average value of 0.26% with a maximum value being 3.39% across different sizes of the supply base. When the cost and non-cost estimates are normally distributed, the gap takes an average value 0.14% with a maximum value being 0.69% across different sizes of the supply base. The computational results indeed verify, for an arbitrary number of contractors and for both uniform and normal beliefs, the attractive performance of the optimal cost-sharing mechanism suggested by Theorem 4.

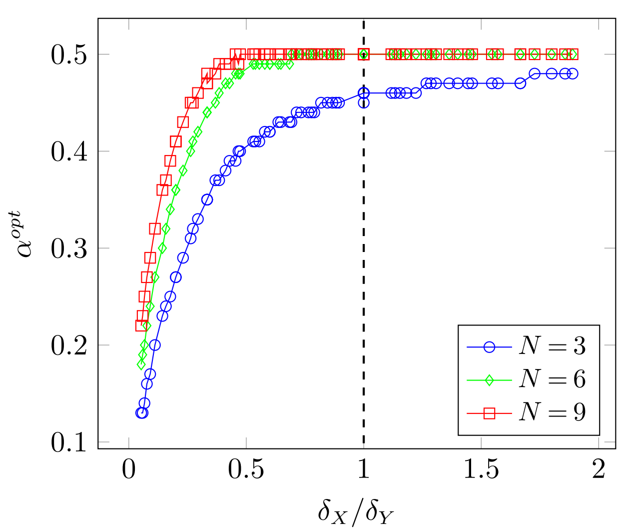
As observed from Table 1, the optimality gap decreases as the number of contractors increases, irrespective of the distributional assumptions on X_n and Y_n. This trend is intuitive and consistent

Number of contractors (N)	$X_n \sim$ Uniform $Y_n \sim$ Uniform		$X_n \sim$ Normal $Y_n \sim$ Normal	
	mean	max	mean	max
2	0.89%	3.39%	0.23%	0.69%
3	0.45%	2.71%	0.19%	0.60%
4	0.28%	2.48%	0.16%	0.57%
5	0.20%	2.25%	0.14%	0.55%
6	0.16%	2.04%	0.13%	0.53%
7	0.13%	1.86%	0.12%	0.52%
8	0.10%	1.69%	0.11%	0.51%
9	0.09%	1.54%	0.11%	0.50%
10	0.07%	1.41%	0.10%	0.49%

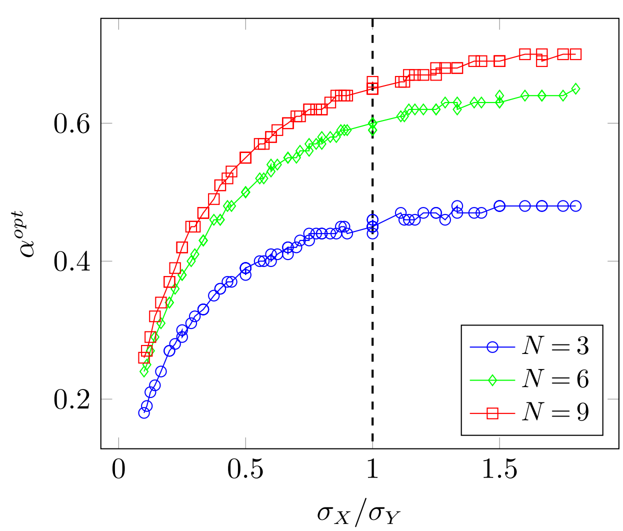
Table 1 **The average and the maximum value of the gap $100[C(\alpha^{opt}) - OPT_{NC}]/OPT_{NC}$ across 100 instances of (δ_X, δ_Y) for uniform distributions and (σ_X, σ_Y) for normal distributions.**

with many other auction settings. As the supply market becomes more competitive with more contractors, the buyer's information rent reduces due to the decrease in the spread between the lowest and the second-lowest order statistics. In turn, this allows the buyer to afford a higher cost-sharing fraction and improve her allocative efficiency. These two favorable effects reduce the buyer's expected cost, thus narrowing the optimality gap.

Optimal Cost-Sharing Fraction. We now turn our attention to the behavior of the optimal cost-sharing fraction α^{opt}. Recall from Theorem 3 that α^{opt} increases as the non-cost estimate Y_n becomes more certain relative to the cost attribute X_n.



(a) $X_n \sim$ Uniform and $Y_n \sim$ Uniform



(b) $X_n \sim$ Normal and $Y_n \sim$ Normal

Figure 2 **A graphical illustration of the optimal cost-sharing parameter α^{opt} for a set of instances in our test-bed.**

In Figure 2(a), for three sizes of the supply base, we compute and plot α^{opt} as a function of δ_X/δ_Y – the behavior is similar to that for $N=2$. We repeat the same exercise in Figure 2(b) for normally distributed X_n and Y_n. We again find α^{opt} to be increasing in σ_X/σ_Y, suggesting that the effect of the relative variability between the cost and non-cost estimates on α^{opt} is robust with respect to the number of contractors.

As Figures 2(a) and 2(b) depict, fixing the distributions of X_n and Y_n, the buyer should choose a higher cost-sharing fraction α^{opt} as the number of contractors increases. As mentioned earlier, a larger supply base reduces the buyer's information rent, which allows her to improve allocative efficiency by increasing the cost-sharing fraction.

5. Endogenous Cost and Non-Cost Attributes

In our analysis thus far, the contractors' cost and non-cost attributes are assumed to be exogenous. However, equally prevalent are situations where contractors can take costly yet unobservable actions to influence their non-cost attributes. For example, in highway construction projects, the winning contractor may hire additional workers to shorten the completion time. In such situations, contractors' cost and non-cost attributes are endogenously determined by their actions. In this section, we demonstrate that our results on cost-sharing mechanisms obtained in the previous section remain robust in such a setting.

We now define this setting. Each contractor $n \in \{1, \cdots, N\}$ can exert an effort $e_n \geq 0$ to reduce his non-cost attribute from y_n to $y_n - e_n$. Such an effort costs him $x_n = g \cdot e_n^2/2$, where $g > 0$ is a publicly-known parameter. The buyer observes neither y_n nor x_n (or equivalently e_n). Once the project is completed, the buyer observes the non-cost attribute $y_n - e_n$ and incurs a disutility cost $w_n = v \cdot (y_n - e_n)^2/2$, where $v > 0$ is also a publicly-known parameter. To rule out the possibility that the buyer does not incentivize any effort even under full information, we assume that $v \geq g$, which implies that the marginal benefit of exerting effort dominates its marginal cost. In this setting, the buyer faces a mixed mechanism-design problem, featuring one-dimensional private information represented by y_n and hidden action represented by e_n. We let y_n be a realization of a uniform random variable Y_n over $[\underline{y}, \overline{y}]$, with $\underline{y} \geq 0$ and Y_n independent across n. For notational convenience, we denote $\delta_Y = \overline{y} - \underline{y}$, $\mathbf{y} = (y_1, y_2, \ldots, y_N)$ and $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_N)$. The functional forms of the cost x_n and the disutility cost w_n, and the choice of the uniform distribution for Y_n have been widely used in the existing literature on incentive contracts (e.g., Rogerson 2003, Chu and Sappington 2007, An and Zhang 2015).

The buyer's objective is to minimize her expected *total cost.*[4] The buyer's problem can be solved by closely following the analysis in Laffont and Tirole (1987). Therefore, for brevity, we only provide a sketch of this analysis in the Electronic Companion (see Lemma D.1 of Appendix D). Denote the buyer's optimal cost by OPT_e. As Rogerson (2003) points out, the optimal mechanism is unattractive from a practical point of view (due to the complex nature of its payment function). Our goal in this section is twofold: (i) to use the optimal mechanism as a theoretical benchmark to demonstrate the near optimality of the cost-sharing mechanism; and (ii) to offer practical guidelines on the choice of the cost-sharing fraction.

Under the cost-sharing mechanism CS_α, each contractor n with type y_n chooses an effort level, denoted by $e^\alpha(y_n)$, to minimize his total cost $x_n + \alpha w_n$. From the buyer's perspective, y_n is a realization of the random variable Y_n; therefore, the contractor's cost and the buyer's ex ante disutility cost can be expressed by the random variables $X_n^\alpha = g \cdot (e^\alpha(Y_n))^2/2$ and $W_n^\alpha = v \cdot (Y_n - e^\alpha(Y_n))^2/2$, respectively. Using arguments similar to those in Lemma 1, one can show that, each contractor n bids his total cost $X_n^\alpha + \alpha W_n^\alpha$; further, this cost is increasing in Y_n. Consequently, the contractor with the lowest Y_n is selected as the winner. Let $Y_{(l)}$ denote the l-th lowest order-statistic of $\{Y_1, \cdots, Y_n\}$. Thus, the buyer's expected cost $C_e(\alpha)$ under the cost-sharing mechanism CS_α deviates from the first-best cost by

$$C_e(\alpha) - \underbrace{\mathbb{E}\left[X_{(1)}^1 + W_{(1)}^1\right]}_{\text{first-best cost}} = \underbrace{\mathbb{E}\left[X_{(1)}^\alpha + W_{(1)}^\alpha - X_{(1)}^1 - W_{(1)}^1\right]}_{\text{effort inefficiency}} + \underbrace{\mathbb{E}\left[X_{(2)}^\alpha + \alpha W_{(2)}^\alpha - X_{(1)}^\alpha - \alpha W_{(1)}^\alpha\right]}_{\text{information rent}}.$$

The above decomposition of the buyer's expected cost is in the same spirit as that in Section 4.1. If the buyer had full information (i.e., knew contractors' types), she would select the contractor with type $Y_{(1)}$ and let him fully reimburse the disutility cost. This would result in an ex ante expected first-best cost of $\mathbb{E}[X_{(1)}^1 + W_{(1)}^1]$ for the buyer. Under the cost-sharing mechanism, the buyer incurs additional costs over and above the first-best cost due to *effort inefficiency* and *information rent.* The effort inefficiency captures the loss caused by the deviation of $e^\alpha(Y_n)$ from its full-information level $e^1(Y_n)$. The information rent represents the surplus given up by the buyer to the selected contractor.

Theorem 5 below establishes the robustness of our earlier results to endogenous cost and non-cost attributes.

Theorem 5. *Under the family of cost-sharing mechanisms $\{CS_\alpha : 0 \le \alpha \le 1\}$, the information rent increases and the effort inefficiency decreases with the cost-sharing fraction α. The optimal* *cost-sharing fraction* α^{opt} *is decreasing in* g/v *and* δ_Y*, but increasing in* N*, ceteris paribus. The buyer's expected cost under the best cost-sharing mechanism exceeds the optimal cost* OPT_e *by no more than 7.2%, i.e.,* $C_e(\alpha^{opt}) \le 1.072 \times OPT_e$*. Moreover,* $C_e(\alpha^{opt}) \to OPT_e$ *as* $N \to \infty$*.*

Consistent with Theorem 1, the first result in Theorem 5 delineates how the cost-sharing fraction α trades off information rent with effort inefficiency. On the one hand, incentivizing a higher effort level calls for a larger α to make the contractor increasingly responsible for the disutility cost. On the other hand, reducing the information rent calls for a smaller α. The optimal fraction α^{opt} strikes this trade-off.

In line with Theorem 3, the second result in Theorem 5 provides the following practical guidelines on the choice of the cost-sharing fraction.

- For larger g/v, the marginal cost of exerting effort becomes higher relative to its marginal benefit, discouraging the buyer from incentivizing a higher effort level. Thus, the buyer chooses a smaller cost-sharing fraction α^{opt} to reduce the information rent.
- As the contractors become more heterogenous in their non-cost attributes (i.e., as δ_Y increases), the information rent increases. To curtail this increase, the buyer chooses a smaller cost-sharing fraction α^{opt}, sacrificing the effort efficiency.
- An increase in the number of contractors naturally intensifies competition, thereby reducing the information rent and allowing the buyer to raise the cost-sharing fraction α^{opt}.

The last two results in Theorem 5 show that the performance of the optimal cost-sharing mechanism continues to be impressive even under endogenous cost and non-cost attributes.

6. Conclusion

We studied the class of auction-based cost-sharing mechanisms for a buyer (firm) who wishes to award a project to one among a set of contractors, in an environment with multi-dimensional asymmetric information. In the setting we analyzed, the buyer faces information asymmetry along two dimensions: each contractor's cost and his estimate of an a priori uncertain non-cost attribute; the buyer observes the winning contractor's realized non-cost attribute, which inflicts on her an associated disutility cost. A cost-sharing mechanism is specified by a single parameter, namely the cost-sharing fraction, which is the percentage of the buyer's realized disutility cost that the winning contractor reimburses upon completion of the project.

Cost-sharing mechanisms are practically appealing in that they solicit only a single bid from each contractor vying for the project and are straightforward to implement. We show that the class of cost-sharing mechanisms is also theoretically well-justified in that a mechanism with a carefully-chosen cost-sharing fraction delivers a near-optimal total cost for the buyer. Our analysis explains the essential trade-off between allocative inefficiency and information rent that the buyer faces under this class of mechanisms, and quantifies the price of anarchy of the optimal cost-sharing mechanism for uniform and normal beliefs. For practitioners, we offer simple recommendations on the choice of the cost-sharing fraction: A buyer who faces a higher uncertainty in the contractors' costs relative to their non-cost estimates, or one who faces a competitive supply base, should use a higher cost-sharing fraction. Finally, we show that our findings hold even when contractors' cost and non-cost attributes are endogenously determined by their effort levels.

There are several directions to enrich cost-sharing mechanisms for other informational environments. For instance, in settings where a non-cost attribute is not readily observable ex-post, the buyer may be able to resort to costly information-acquisition instruments such as audits to obtain an ex-post verifiable signal on that attribute, and assess the value of such audits while restricting attention to the family of cost-sharing mechanisms. In settings with multiple ex-post observable non-cost attributes, one can extend our cost-sharing mechanisms to attach (possibly different) cost-sharing fractions to each dimension. In practice (especially in awarding public projects such as the construction of roads), the buyer (e.g., government) tends to be budget-constrained and needs to partner with (sometimes multiple) contractors to raise sufficient funds for the project. In such cases, the buyer may consider offering concession contracts with a revenue-sharing provision instead of a cost-sharing clause. While cost-sharing mechanisms involve contingent payments, one can also envision other types of contingent clauses. For instance, original equipment manufacturers typically include in their supply contracts contingent clauses for future businesses or supplier-development investments based on the quality (a non-cost attribute) delivered by their suppliers.

Endnotes

1. One could argue that the winning contractor can exert costly effort to expedite the completion time of the project, if required. This feature is considered in Section 5.

2. Typically lane-rental mechanisms select a contractor using the first-price sealed-bid auction. Given that contractors submit single-dimensional bids, by the Revenue Equivalence Theorem (see e.g., Krishna 2009), the first-price sealed-bid and second-price sealed-bid auctions are revenue equivalent for the buyer.

3. This is a standard assumption in mechanism design and covers a wide range of commonly-used probability distributions (Bagnoli and Bergstrom 2005).

4. As in Section 3, we will allow for the possibility of the winning contractor deliberately inflating the buyer's observation of his non-cost attribute. In what follows, we will solve a relaxed problem that ignores this possibility. The optimal mechanism to this relaxed problem is free of such manipulation and is, therefore, also optimal for the problem that allows the possibility of such an inflation.

Appendix A: Auxiliary Results

The proofs of the lemmas below are relegated to Appendix C in the Electronic Companion.

LEMMA A.1. *For class $\mathcal{P}_U$, the cost $C(\alpha)$ is given by*

$$C(\alpha)=\begin{cases}\mu_X+\mu_W+\frac{\delta_X}{6}-\frac{\delta_W^2}{6\delta_X}\alpha+\frac{\delta_W^2}{4\delta_X}\alpha^2+\frac{\delta_W^3}{20\delta_X^2}\alpha^2-\frac{\delta_W^3}{15\delta_X^2}\alpha^3, & \textit{if} \quad \delta_X\geq\alpha\delta_W,\\ \mu_X+\mu_W-\frac{\delta_W}{6}+\frac{\delta_W}{3}\alpha+\frac{\delta_X^2}{12\delta_W}\frac{1}{\alpha^2}+\frac{\delta_X^3}{60\delta_W^2}\frac{1}{\alpha^2}-\frac{\delta_X^3}{30\delta_W^2}\frac{1}{\alpha^3}, & \textit{if} \quad \delta_X\leq\alpha\delta_W.\end{cases} \tag{A.1}$$

LEMMA A.2. *For class $\mathcal{P}_U$, the optimal cost-sharing fraction α^{opt} is given by the solution of*

$$\frac{1}{3}-\frac{1}{6\alpha^3}\left(\frac{\delta_X}{\delta_W}\right)^2-\frac{1}{30\alpha^3}\left(\frac{\delta_X}{\delta_W}\right)^3+\frac{1}{10\alpha^4}\left(\frac{\delta_X}{\delta_W}\right)^3=0, \tag{A.2}$$

which we show belongs in $[\delta_X/\delta_W,1]$ when $\delta_X/\delta_W\leq 2/9$, and is given by

$$\alpha^{opt}=\left(\frac{1}{4}+\frac{5\delta_X}{4\delta_W}\right)-\sqrt{\left(\frac{1}{4}+\frac{5\delta_X}{4\delta_W}\right)^2-\frac{5\delta_X}{6\delta_W}}\in[0,\delta_X/\delta_W], \tag{A.3}$$

when $\delta_X/\delta_W\geq 2/9$.

LEMMA A.3. *For class $\mathcal{P}_N$, the cost $C(\alpha)$ is given by:*

$$C(\alpha)=\mu_X+\mu_W+\frac{\sigma_W}{\sqrt{\pi}}\left[\sqrt{\frac{\sigma_X^2}{\sigma_W^2}+\alpha^2}-\frac{\alpha(1-\alpha)}{\sqrt{\frac{\sigma_X^2}{\sigma_W^2}+\alpha^2}}\right]. \tag{A.4}$$

LEMMA A.4. *For class $\mathcal{P}_N$, the optimal cost-sharing fraction, α^{opt} is obtained from solving*

$$\frac{2\sigma_W^2}{\sigma_X^2}\alpha^3+3\alpha-1=0. \tag{A.5}$$

LEMMA A.5. *Assume $\delta_X/\delta_W\geq 1/2$ and fix an $\alpha\in[0,1]$ such that $\delta_X\geq\alpha\delta_W$. Let $\delta_{W/X}=\delta_W/\delta_X$. Then, $[C(\alpha)-OPT_{NC}]/OPT_{NC}=h(\alpha,\delta_{W/X})\left(\frac{1}{6}-\frac{\delta_{W/X}^2}{24}+\frac{\delta_{W/X}^3}{240}+\frac{\mu_X+\mu_W}{\delta_X}\right)^{-1}$, where $h(\alpha,\delta_{W/X}):=\delta_{W/X}^2(-\alpha/6+\alpha^2/4+\alpha^2\delta_{W/X}/20-\alpha^3\delta_{W/X}/15+1/24-\delta_{W/X}/240)$, is increasing $\delta_{W/X}$ and δ_X.*

LEMMA A.6. *Assume $\delta_X\in[\mu_W,2\mu_X]$, $\delta_W=2\mu_W$ and fix an $\alpha\in[0,1]$ such that $\delta_X\geq\alpha\delta_W$. Then, $[C(\alpha)-OPT_{NC}]/OPT_{NC}=l(\alpha,\delta_{W/X})\left(\frac{1}{3\delta_{W/X}}-\frac{\delta_{W/X}}{12}+\frac{\delta_{W/X}^2}{120}+\frac{\mu_X}{\mu_W}+1\right)^{-1}$, where $l(\alpha,\delta_{W/X}):=-\frac{\alpha\delta_{W/X}}{3}+\frac{\alpha^2\delta_{W/X}}{2}+\frac{\alpha^2\delta_{W/X}^2}{10}-\frac{2\alpha^3\delta_{W/X}^2}{15}+\frac{\delta_{W/X}}{12}-\frac{\delta_{W/X}^2}{120}$, is increasing in $\delta_{W/X}$.*

Appendix B: Proofs of Results in Section 4

Proof of Theorem 2. We first establish the result for Class $\mathcal{P}_U$. It is straightforward to show that

$$FB = \begin{cases} \mu_X + \mu_W - \frac{\delta_X}{6} - \frac{\delta_W^2}{12\delta_X} + \frac{\delta_W^3}{60\delta_X^2} & \text{if} \quad \delta_X \geq \delta_W, \\ \mu_X + \mu_W - \frac{\delta_W}{6} - \frac{\delta_X^2}{12\delta_W} + \frac{\delta_X^3}{60\delta_W^2} & \text{if} \quad \delta_X \leq \delta_W. \end{cases} \tag{B.1}$$

Let $\delta_r = \delta_X/\delta_W$. We first consider $\delta_r \leq 2/9$. Since $C(\alpha^{opt}) \leq C(\alpha)|_{\alpha=3\delta_r/2}$, we have:

$$\frac{C(\alpha^{opt}) - FB}{FB} \leq \frac{C(3\delta_r/2) - FB}{FB} = \left.\frac{\frac{\alpha}{3} + \frac{1}{12}\frac{\delta_r^2}{\alpha^2} + \frac{1}{60}\frac{\delta_r^3}{\alpha^2} - \frac{1}{30}\frac{\delta_r^3}{\alpha^3} + \frac{\delta_r^2}{12} - \frac{\delta_r^3}{60}}{\frac{\mu_X+\mu_W}{\delta_W} - \frac{1}{6} - \frac{\delta_r^2}{12} + \frac{\delta_r^3}{60}}\right|_{\alpha=3\delta_r/2}$$

$$\leq \left.\frac{\frac{\alpha}{3} + \frac{1}{12}\frac{\delta_r^2}{\alpha^2} + \frac{1}{60}\frac{\delta_r^3}{\alpha^2} - \frac{1}{30}\frac{\delta_r^3}{\alpha^3} + \frac{\delta_r^2}{12} - \frac{\delta_r^3}{60}}{\frac{\delta_r}{2} + \frac{1}{2} - \frac{1}{6} - \frac{\delta_r^2}{12} + \frac{\delta_r^3}{60}}\right|_{\alpha=3\delta_r/2}$$

where the last inequality follows from the observation that $C(\alpha) \geq FB$ for all α, and our assumptions that $\underline{x} \geq 0$ and $\underline{w} \geq 0$ (which also imply $FB \geq 0$). With algebraic simplification, we obtain: $\frac{C(\alpha^{opt})-FB}{FB} \leq g(\delta_r) = \frac{146/405+136\delta_r/135}{1/3+\delta_r/2-\delta_r^2/12+\delta_r^3/60} - 1$. To obtain an upper bound on $g(\delta_r)$, we show that it is increasing in δ_r, and hence its maxima is found at $\delta_r = 2/9$. Let $\frac{d}{d\delta_r}g(\delta_r) = \frac{1}{(1/3+\delta_r/2-\delta_r^2/12+\delta_r^3/60)^2}h(\delta_r)$, where $h(\delta_r) = \frac{7}{45} + \frac{73\delta_r}{1215} + \frac{267\delta_r^2}{4050} - \frac{68\delta_r^3}{2025}$. It is fairly easy to show that $\frac{d}{d\delta_r}h(\delta_r)$ is increasing in $\delta_r \in [0, 2/9]$ and is always non-negative. Thus, $h(\delta_r)$ is increasing in $\delta_r \in [0, 2/9]$. Further, $h(0) = 7/45 > 0$. Clearly, $g(\delta_r)$ is increasing in δ_r with the maximum value of 0.3266 at $\delta_r = 2/9$. Thus, for $\delta_r \leq 2/9$, our result is summarized as follows:

$$\frac{C(\alpha^{opt}) - FB}{FB} \leq 0.3266. \tag{B.2}$$

We next consider $\delta_r \in [2/9, 1]$. Since $C(\alpha^{opt}) \leq C(\alpha)|_{\alpha=0}$, we have:

$$\frac{C(\alpha^{opt}) - FB}{FB} \leq \frac{C(0) - FB}{FB}$$

$$= \left.\frac{\frac{\delta_r}{6} - \frac{1}{6}\frac{\alpha}{\delta_r} + \frac{1}{4}\frac{\alpha^2}{\delta_r} + \frac{1}{20}\frac{\alpha^2}{\delta_r^2} - \frac{1}{15}\frac{\alpha^3}{\delta_r^2} + \frac{1}{6} + \frac{\delta_r^2}{12} - \frac{\delta_r^3}{60}}{\frac{\mu_X+\mu_W}{\delta_W} - \frac{1}{6} - \frac{\delta_r^2}{12} + \frac{\delta_r^3}{60}}\right|_{\alpha=0} \leq \left.\frac{\frac{\delta_r}{6} - \frac{1}{6}\frac{\alpha}{\delta_r} + \frac{1}{4}\frac{\alpha^2}{\delta_r} + \frac{1}{20}\frac{\alpha^2}{\delta_r^2} - \frac{1}{15}\frac{\alpha^3}{\delta_r^2} + \frac{1}{6} + \frac{\delta_r^2}{12} - \frac{\delta_r^3}{60}}{\frac{\delta_r}{2} + \frac{1}{2} - \frac{1}{6} - \frac{\delta_r^2}{12} + \frac{\delta_r^3}{60}}\right|_{\alpha=0}.$$

With algebraic simplifications, we obtain: $\frac{C(\alpha^{opt})-FB}{FB} \leq g(\delta_r) = \frac{1/2+2\delta_r/3}{1/3+\delta_r/2-\delta_r^2/12+\delta_r^3/60} - 1$. To obtain an upper bound on $g(\delta_r)$, we first show that it is a quasi-convex in δ_r, and hence its maxima is found at the two extreme points over the interval $\delta_r \in [2/9, 1]$. Let $\frac{d}{d\delta_r}g(\delta_r) = \frac{1}{(1/3+\delta_r/2-\delta_r^2/12+\delta_r^3/60)^2}h(\delta_r)$, where $h(\delta_r) = -\frac{1}{36} + \frac{\delta_r}{12} + \frac{11\delta_r^2}{360} - \frac{2\delta_r^3}{90}$. It is fairly easy to show that $\frac{d}{d\delta_r}h(\delta_r)$ is non-negative for $\delta_r \in [2/9, 1]$. Thus, $h(\delta_r)$ is increasing in δ_r. Further, $h(2/9) = -0.008$ and $h(1) = 0.064$. Clearly, $g(\delta_r)$ is quasi-convex in δ_r. Further, $g(2/9) = 0.4714$ and $g(1) = 0.5217$. Thus, for $\delta_r \in [2/9, 1]$, our result is summarized as follows:

$$\frac{C(\alpha^{opt}) - FB}{FB} \leq 0.5217. \tag{B.3}$$

Finally, we consider $\delta_r \geq 1$. Since $C(\alpha^{opt}) \leq C(\alpha)|_{\alpha=0}$, we have:

$$\frac{C(\alpha^{opt}) - FB}{FB} \leq \frac{C(0) - FB}{FB}$$

$$= \left.\frac{\frac{\delta_r}{3} - \frac{1}{6}\frac{\alpha}{\delta_r} + \frac{1}{4}\frac{\alpha^2}{\delta_r} + \frac{1}{20}\frac{\alpha^2}{\delta_r^2} - \frac{1}{15}\frac{\alpha^3}{\delta_r^2} + \frac{1}{12}\frac{1}{\delta_r} - \frac{1}{60}\frac{1}{\delta_r^2}}{\frac{\mu_X+\mu_W}{\delta_W} - \frac{\delta_r}{6} - \frac{1}{12}\frac{1}{\delta_r} + \frac{1}{60}\frac{1}{\delta_r^2}}\right|_{\alpha=0} \leq \left.\frac{\frac{\delta_r}{3} - \frac{1}{6}\frac{\alpha}{\delta_r} + \frac{1}{4}\frac{\alpha^2}{\delta_r} + \frac{1}{20}\frac{\alpha^2}{\delta_r^2} - \frac{1}{15}\frac{\alpha^3}{\delta_r^2} + \frac{1}{12}\frac{1}{\delta_r} - \frac{1}{60}\frac{1}{\delta_r^2}}{\frac{\delta_r}{2} + \frac{1}{2} - \frac{\delta_r}{6} - \frac{1}{12}\frac{1}{\delta_r} + \frac{1}{60}\frac{1}{\delta_r^2}}\right|_{\alpha=0}.$$

With algebraic simplifications, we obtain: $\frac{C(\alpha^{opt})-FB}{FB} \le g(\delta_r) = \frac{1/2+2\delta_r/3}{1/2+\delta_r/3-1/(12\delta_r)+1/(60\delta_r^2)} - 1$. To obtain an upper bound on $g(\delta_r)$, we show that it is an increasing in $\delta_r \ge 1$. We have: $\frac{d}{d\delta_r} g(\delta_r) = \frac{1}{(1/2+\delta_r/3-1/(12\delta_r)+1/(60\delta_r^2))^2} h(\delta_r)$, where $h(\delta_r) = \frac{1}{6} - \frac{1}{9\delta_r} - \frac{1}{120\delta_r^2} + \frac{1}{60\delta_r^3}$. It is fairly easy to show that $h(\delta_r)$ is increasing in $\delta_r \ge 1$. Further, $h(1) = 23/360 > 0$; thus $g(\delta_r)$ is increasing in δ_r, and attains a maximum value of 1 in the limit $\delta_r \to \infty$. Thus, for $\delta_r \ge 1$, our result is summarized as follows:

$$\frac{C(\alpha^{opt})-FB}{FB} \le 1. \tag{B.4}$$

Combining our results from (B.2)–(B.4) establishes the upper bound of 2 for the ratio $C(\alpha^{opt})/FB$ for the class $\mathcal{P}_U$. We now show that this bound is tight when $\mu_X = \delta_X/2$, $\mu_W = \delta_W/2$ and $\delta_W \to 0$ (so that $\delta_X/\delta_W \to \infty$).

From Lemma A.2, it is easy to see that $\alpha^{opt} \to 1/3$ as $\delta_r \to \infty$. Using $\mu_X = \delta_X/2$ and $\mu_W = \delta_W/2$, we have

$$\lim_{\delta_r\to\infty} \frac{C(\alpha^{opt})-FB}{FB} = \lim_{\delta_r\to\infty}\left[\frac{\frac{1}{3} - \frac{1}{6}\frac{\alpha}{\delta_r^2} + \frac{1}{4}\frac{\alpha^2}{\delta_r^2} + \frac{1}{20}\frac{\alpha^2}{\delta_r^3} - \frac{1}{15}\frac{\alpha^3}{\delta_r^3} + \frac{1}{12}\frac{1}{\delta_r^2} - \frac{1}{60}\frac{1}{\delta_r^3}}{\frac{1}{2} + \frac{1}{2\delta_r} - \frac{1}{6} - \frac{1}{12}\frac{1}{\delta_r^2} + \frac{1}{60}\frac{1}{\delta_r^3}} \Bigg|_{\alpha=\alpha^{opt}} \right] = \frac{1/3}{1/2-1/6} = 1.$$

We now consider <u>Class $\mathcal{P}_N$</u>. It is easy to show that $FB = \mathbb{E}[X_{1(1)} + W_{1(1)}] = \mu_X + \mu_W - \frac{\sqrt{\sigma_X^2+\sigma_W^2}}{\sqrt{\pi}}$. Let $\sigma_r = \sigma_X/\sigma_W$. When $\mu_X \ge 3\sigma_X$ and $\mu_W \ge 3\sigma_W$, we have $\sqrt{\pi}(\mu_X+\mu_W)/\sigma_W - \sqrt{1+\sigma_r^2} \ge 3\sqrt{\pi}(1+\sigma_r) - \sqrt{(1+\sigma_r^2)} > 0$. Thus, $FB > 0$. Also, $C(\alpha) \ge FB$ for all α. Then,

$$\frac{C(\alpha^{opt})-FB}{FB} \le \frac{C(0)-FB}{FB} = \frac{\sqrt{\alpha^2+\sigma_r^2} - \frac{\alpha(1-\alpha)}{\sqrt{\alpha^2+\sigma_r^2}} + \sqrt{1+\sigma_r^2}}{\sqrt{\pi}(\mu_X+\mu_W)/\sigma_W - \sqrt{1+\sigma_r^2}}\Bigg|_{\alpha=0}$$

$$\le \frac{\sqrt{\alpha^2+\sigma_r^2} - \frac{\alpha(1-\alpha)}{\sqrt{\alpha^2+\sigma_r^2}} + \sqrt{1+\sigma_r^2}}{3\sqrt{\pi}(1+\sigma_r) - \sqrt{1+\sigma_r^2}}\Bigg|_{\alpha=0} = g(\sigma_r) := \frac{\sigma_r + \sqrt{1+\sigma_r^2}}{3\sqrt{\pi}(1+\sigma_r) - \sqrt{1+\sigma_r^2}}.$$

Differentiating $g(\sigma_r)$, we obtain:

$$\frac{d}{d\sigma_r} g(\sigma_r) = \frac{1}{\left(3\sqrt{\pi}(1+\sigma_r) - \sqrt{1+\sigma_r^2}\right)^2} \frac{1}{\sqrt{1+\sigma_r^2}} \times \left[3\sqrt{\pi}\left(\sqrt{1+\sigma_r^2} + \sigma_r - 1\right) - 1\right].$$

Clearly, $3\sqrt{\pi}\left(\sqrt{1+\sigma_r^2}+\sigma_r-1\right) - 1$ is increasing in σ_r, taking a negative value at $\sigma_r = 0$ and a positive value at $\sigma_r > \frac{(1+1/(3\sqrt{\pi}))^2-1}{2+2/(3\sqrt{\pi})}$; thus, $g(\sigma_r)$ is quasi-convex in σ_r, and its maximum value will be achieved either at $\sigma_r = 0$ or $\sigma_r \to \infty$. Further, $g(0) = 1/(3\sqrt{\pi}-1)$ and $g(\infty) = 2/(3\sqrt{\pi}-1)$. Therefore, $C(\alpha^{opt})/FB \le (3\sqrt{\pi}+1)/(3\sqrt{\pi}-1)$. We now show that this bound is tight when $\mu_X = 3\sigma_X$, $\mu_W = 3\sigma_W$ and $\sigma_W \to 0$. It is easy to see that $\alpha^{opt} \to 1/3$ as $\sigma_r \to \infty$. Next,

$$\lim_{\sigma_r\to\infty} \frac{C(\alpha^{opt})-FB}{FB} = \lim_{\sigma_r\to\infty}\left[\frac{\sqrt{\alpha^2/\sigma_r^2+1} - \frac{\alpha(1-\alpha)}{\sigma_r\sqrt{\alpha^2+\sigma_r^2}} + \sqrt{1/\sigma_r^2+1}}{3\sqrt{\pi}(1/\sigma_r+1) - \sqrt{1/\sigma_r^2+1}}\Bigg|_{\alpha=\alpha^{opt}}\right] = \frac{2}{3\sqrt{\pi}-1}.$$

This completes the proof of Theorem 2. ■

Proof of Theorem 3. We first consider the <u>Class $\mathcal{P}_U$</u>. We use Lemmas A.1 and A.2 to prove that α^{opt} is increasing in δ_X/δ_W. For ease of notation, let $\delta_r = \delta_X/\delta_W$. Consider the case when $\delta_r \le 2/9$. Then, from (A.2), $\alpha^{opt} = \alpha(\delta_r) \in [\delta_r, 1]$. Rewriting (A.2), we get: $\frac{\alpha^4(\delta_r)}{3} - \frac{\alpha(\delta_r)}{6}\delta_r^2(1+\delta_r/5) + \frac{\delta_r^3}{10} = 0$. Then, $\frac{d}{d\delta_r}\alpha(\delta_r) = \frac{\alpha\delta_r/3+(\alpha-3)\delta_r^2/10}{4\alpha^3/3-(\delta_r^2/6)(1+\delta_r/5)}\Big|_{\alpha=\alpha(\delta_r)}$. Since $\alpha(\delta_r) \ge \delta_r$, the numerator is $\frac{\delta_r}{3}\alpha(\delta_r) + \frac{\delta_r^2}{10}(\alpha(\delta_r)-3) \ge \frac{\delta_r^2}{3} + \frac{\delta_r^2(\delta_r-3)}{10} =$

$\frac{\delta_r^3}{10}+\frac{\delta_r^2}{30}>0$. Thus, to prove that $\alpha(\delta_r)$ is increasing in δ_r, it remains to show that the denominator $4\alpha^3/3-(\delta_r^2/6)(1+\delta_r/5)$ is also greater than 0 at $\alpha=\alpha(\delta_r)$. This is shown as follows:

$$\alpha(\delta_r)\left(\frac{4\alpha^3(\delta_r)}{3}-\frac{\delta_r^2(1+\delta_r/5)}{6}\right)=\frac{4\alpha^4(\delta_r)}{3}-\frac{\delta_r^2(1+\delta_r/5)}{6}\alpha(\delta_r)=\frac{\delta_r^2(1+\delta_r/5)}{2}\alpha(\delta_r)-\frac{2\delta_r^3}{5}$$
$$\geq\frac{\delta_r^3(1+\delta_r/5)}{2}-\frac{2\delta_r^3}{5}=\frac{\delta_r^3}{10}+\frac{\delta_r^4}{10}>0, \tag{B.5}$$

where the second equality follows from applying the definition of $\alpha(\delta_r)$ and (B.5) follows from the observation that $\alpha(\delta_r)\geq\delta_r>0$. This completes the case $\delta_r\leq 2/9$.

Next, we consider the case $\delta_r\geq 2/9$. Taking the first-order derivative of (A.3) with respect to δ_r yields

$$\frac{d}{d\delta_r}\alpha(\delta_r)=\frac{5}{4}-\frac{(25\delta_r-5/3)}{16}\left[\left(\frac{1}{4}+\frac{5}{4}\delta_r\right)^2-\frac{5}{6}\delta_r\right]^{-1/2},$$

which can be easily shown to be positive for any δ_r. This completes the proof for the class $\mathcal{P}_U$. □

We now consider the Class $\mathcal{P}_N$ and use Lemma A.4. Differentiating the first-order condition in (A.5) with respect to σ_W/σ_X, we obtain: $\frac{d}{d(\sigma_W/\sigma_X)}\alpha^{opt}=-\frac{4\sigma_W}{3\sigma_X}\times\frac{(\alpha^{opt})^3}{1+2(\alpha^{opt}\sigma_W/\sigma_X)^2}\leq 0$. Thus, α^{opt} is increasing in σ_X/σ_W. This completes the proof of Theorem 3. ■

Proof of Theorem 4. For Class $\mathcal{P}_U$, we have $OPT_{NC}=2\mathbb{E}\left[X_{1(1/2)}+\frac{W_{1(1/2)}}{2}\right]-\underline{x}$, which can be easily calculated as

$$OPT_{NC}=\begin{cases}\mu_X+\mu_W+\frac{\delta_X}{2}-\frac{\delta_W}{6}-\frac{\delta_X^2}{3\delta_W}+\frac{2\delta_X^3}{15\delta_W^2}, & \text{if } \delta_X/\delta_W\leq 1/2,\\ \mu_X+\mu_W+\frac{\delta_X}{6}-\frac{1}{24}\frac{\delta_W^2}{\delta_X}+\frac{1}{240}\frac{\delta_W^3}{\delta_X^2}, & \text{if } \delta_X/\delta_W\geq 1/2.\end{cases} \tag{B.6}$$

Let $\delta_r=\delta_X/\delta_W$. We first consider $\delta_r\leq 2/9$. From Lemma A.2, we know that $\alpha^{opt}\in[\delta_r,1]$. Since $C(\alpha^{opt})\leq C(\alpha)|_{\alpha=3\delta_r/2}$, we have:

$$\frac{C(\alpha^{opt})-OPT_{NC}}{OPT_{NC}}\leq\frac{C(3\delta_r/2)-OPT_{NC}}{OPT_{NC}}$$
$$=\left.\frac{\frac{\alpha}{3}+\frac{1}{12}\frac{\delta_r^2}{\alpha^2}+\frac{1}{60}\frac{\delta_r^3}{\alpha^2}-\frac{1}{30}\frac{\delta_r^3}{\alpha^3}-\frac{\delta_r}{2}+\frac{\delta_r^2}{3}-\frac{2\delta_r^3}{15}}{\frac{\mu_X+\mu_W}{\delta_W}+\frac{\delta_r}{2}-\frac{1}{6}-\frac{\delta_r^2}{3}+\frac{2\delta_r^3}{15}}\right|_{\alpha=3\delta_r/2}\leq\left.\frac{\frac{\alpha}{3}+\frac{1}{12}\frac{\delta_r^2}{\alpha^2}+\frac{1}{60}\frac{\delta_r^3}{\alpha^2}-\frac{1}{30}\frac{\delta_r^3}{\alpha^3}-\frac{\delta_r}{2}+\frac{\delta_r^2}{3}-\frac{2\delta_r^3}{15}}{\frac{\delta_r}{2}+\frac{1}{2}+\frac{\delta_r}{2}-\frac{1}{6}-\frac{\delta_r^2}{3}+\frac{2\delta_r^3}{15}}\right|_{\alpha=3\delta_r/2}.$$

With algebraic simplifications, we obtain: $\frac{C(\alpha^{opt})-OPT_{NC}}{OPT_{NC}}\leq g(\delta_r)=\frac{11/405+\delta_r/135+\delta_r^2/3-2\delta_r^3/15}{1/3+\delta_r-\delta_r^2/3+2\delta_r^3/15}$. To obtain an upper bound on $g(\delta_r)$, we show that it is quasi-convex in δ_r, and hence its maxima is found at the extreme points over the interval $\delta_r\in[0,2/9]$. Let $\frac{d}{d\delta_r}g(\delta_r)=\frac{1}{(1/3+\delta_r-\delta_r^2/3+2\delta_r^3/15)^2}h(\delta_r)$, where $h(\delta_r)=-\frac{2}{81}+\frac{292\delta_r}{1215}+\frac{388\delta_r^2}{2025}-\frac{544\delta_r^3}{2025}$. It is fairly easy to show that $\frac{d}{d\delta_r}h(\delta_r)$ is increasing in $\delta_r\in[0,2/9]$ and is always non-negative. Thus, $h(\delta_r)$ is increasing in $\delta_r\in[0,2/9]$. Further, $h(0)=-0.0247$ and $h(2/9)=0.0352$. Clearly, $g(\delta_r)$ is quasi-convex in δ_r. Further, $g(0)=0.0815$ and $g(2/9)=0.0810$. Thus, for $\delta_r\leq 2/9$, our result is summarized as follows:

$$\frac{C(\alpha^{opt})-OPT_{NC}}{OPT_{NC}}\leq 0.0815. \tag{B.7}$$

We next consider $\delta_r\in[2/9,1/2]$. From Lemma A.2, we know that $\alpha^{opt}\in[0,\delta_r]$. Since $C(\alpha^{opt})\leq C(\alpha)|_{\alpha=\delta_r}$, we have:

$$\frac{C(\alpha^{opt})-OPT_{NC}}{OPT_{NC}}\leq\frac{C(\delta_r)-OPT_{NC}}{OPT_{NC}}=\left.\frac{\frac{\delta_r}{6}-\frac{1}{6}\frac{\alpha}{\delta_r}+\frac{1}{4}\frac{\alpha^2}{\delta_r}+\frac{1}{20}\frac{\alpha^2}{\delta_r^2}-\frac{1}{15}\frac{\alpha^3}{\delta_r^2}-\frac{\delta_r}{2}+\frac{1}{6}+\frac{\delta_r^2}{3}-\frac{2\delta_r^3}{15}}{\frac{\mu_X+\mu_W}{\delta_W}+\frac{\delta_r}{2}-\frac{1}{6}-\frac{\delta_r^2}{3}+\frac{2\delta_r^3}{15}}\right|_{\alpha=\delta_r}$$
$$\leq\left.\frac{\frac{\delta_r}{6}-\frac{1}{6}\frac{\alpha}{\delta_r}+\frac{1}{4}\frac{\alpha^2}{\delta_r}+\frac{1}{20}\frac{\alpha^2}{\delta_r^2}-\frac{1}{15}\frac{\alpha^3}{\delta_r^2}-\frac{\delta_r}{2}+\frac{1}{6}+\frac{\delta_r^2}{3}-\frac{2\delta_r^3}{15}}{\frac{\delta_r}{2}+\frac{1}{2}+\frac{\delta_r}{2}-\frac{1}{6}-\frac{\delta_r^2}{3}+\frac{2\delta_r^3}{15}}\right|_{\alpha=\delta_r}.$$

With algebraic simplification, we obtain: $\frac{C(\alpha^{opt})-OPT_{NC}}{OPT_{NC}} \le g(\delta_r) = \frac{1/20-3\delta_r/20+\delta_r^2/3-2\delta_r^3/15}{1/3+\delta_r-\delta_r^2/3+2\delta_r^3/15}$. To obtain an upper bound on $g(\delta_r)$, we show that it is a quasi-convex function and hence its maxima is found at the two extreme points over the interval $\delta_r \in [2/9, 1/2]$. Let $\frac{d}{d\delta_r} g(\delta_r) = \frac{1}{(1/3+\delta_r-\delta_r^2/3+2\delta_r^3/15)^2} h(\delta_r)$, where $h(\delta_r) = -\frac{1}{10} + \frac{23\delta_r}{90} + \frac{13\delta_r^2}{100} - \frac{17\delta_r^3}{75}$. It is fairly easy to show that $\frac{d}{d\delta_r} h(\delta_r)$ is decreasing in $\delta_r \in [2/9, 1/2]$ and is always non-negative. Thus, $h(\delta_r)$ is increasing in $\delta_r \in [2/9, 1/2]$. Further, $h(2/9) = -0.039$ and $h(1/2) = 0.319$. Clearly, $g(\delta_r)$ is quasi-convex in δ_r. Further, $g(2/9) = 0.059$ and $g(1/2) = 0.054$. Thus, for $\delta_r \in [2/9, 1/2]$, our result is summarized as follows:

$$\frac{C(\alpha^{opt}) - OPT_{NC}}{OPT_{NC}} \le 0.059. \tag{B.8}$$

We now consider $\delta_r \ge 1/2$. Let $\delta_{W/X} = \delta_W/\delta_X = 1/\delta_r$. We know that $\alpha^{opt} \in [0, \delta_r]$ and is given by (A.3). We fix an arbitrary α such that $\delta_X \ge \alpha\delta_W$. We maximize $(C(\alpha) - OPT_{NC})/OPT_{NC}$ with respect to $\delta_{W/X}$ and δ_X, where $\delta_{W/X}$ and δ_X belongs to the set $\{\delta_{W/X} \in [0,2], \delta_X \in [0, 2\mu_X] \mid \delta_{W/X}\delta_X \le 2\mu_W\}$. We consider the following cases: **Case 1**: $\mu_X > \mu_W/2$: From Lemma A.5, we know that $[C(\alpha) - OPT_{NC}]/OPT_{NC}$ is increasing in $\delta_{W/X}$ and δ_X. Given the case assumption, it is easy to see that its maximum value is achieved when $\delta_{W/X}\delta_X = 2\mu_W$ (i.e., $\delta_W = 2\mu_W$). Substituting this condition, Lemma A.6 expresses the relative gap $[C(\alpha) - OPT_{NC}]/OPT_{NC}$ as an increasing function of $\delta_{W/X}$, which is maximized by setting $\delta_{W/X} = 2$; in other words, by setting $\delta_X = \mu_W < 2\mu_X$ and $\delta_W = 2\mu_W$. **Case 2**: $\mu_X \le \mu_W/2$: From Lemma A.5, we know that $[C(\alpha) - OPT_{NC}]/OPT_{NC}$ is increasing in $\delta_{W/X}$ and δ_X. Given the case assumption, it is easy to see that its maximum value is achieved by setting $\delta_{W/X} = 2$ and $\delta_X = 2\mu_X$; in other words, by using $\delta_X = 2\mu_X$ and $\delta_W = 4\mu_X \le 2\mu_W$. From the above two cases, to obtain the maximum value of $(C(\alpha) - OPT_{NC})/OPT_{NC}$, we should set $\delta_{W/X} = 2$ and $\delta_X = \min(2\mu_X, \mu_W)$. Thus, we obtain:

$$\begin{aligned}\frac{C(\alpha) - OPT_{NC}}{OPT_{NC}} &= h(\alpha, \delta_{W/X}) \left(\frac{1}{6} - \frac{\delta_{W/X}^2}{24} + \frac{\delta_{W/X}^3}{240} + \frac{\mu_X + \mu_W}{\delta_X} \right)^{-1} \\ &\le h(\alpha, 2) \left(\frac{1}{6} - \frac{4}{24} + \frac{8}{240} + \frac{\mu_X + \mu_W}{\min(2\mu_X, \mu_W)} \right)^{-1} \le h(\alpha, 2) \left(\frac{1}{30} + \frac{3}{2} \right)^{-1} = \frac{15}{23} h(\alpha, 2).\end{aligned}$$

The above relationship is true for any $\alpha \in [0, \delta_r]$. When $\delta_{W/X} = 2$, (A.3) implies that $\alpha^{opt} \in [0, \delta_r]$ takes the value 0.2843, which results in

$$\frac{C(\alpha^{opt}) - OPT_{NC}}{OPT_{NC}} \le \frac{15}{23} h(0.2843, 2) = 0.0292. \tag{B.9}$$

Our results in (B.7)–(B.9) complete the proof of Theorem 4. ■

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Electronic Companion – *Procurement with Cost and Non-Cost Attributes: Cost-Sharing Mechanisms*

Appendix C: Proofs of Lemma 1 and Lemmas in Appendix A

Proof of Lemma 1: The proof of this result mimics the standard analysis of second-price sealed-bid auctions. Fix an arbitrary α. Consider an arbitrary contractor $n \in \mathcal{N}$ and a bid vector $\mathbf{b}_{-n}$ for all contractors $m \neq n$. We consider the following two cases: **Case 1**: $x_n + \alpha\mathbb{E}_\epsilon[V(y_n+\epsilon)] \leq \min_{m\neq n} b_m$: If contractor n bids truthfully (i.e., submits a bid that is equal to his true total cost), then he obtains a positive utility given by $\min_{m\neq n} b_m - x_n - \alpha\mathbb{E}_\epsilon[V(y_n+\epsilon)]$. We now argue that, for contractor n, bidding truthfully weakly dominates all other strategies. If the contractor bids $b_n \leq x_n + \mathbb{E}_\epsilon[V(y_n+\epsilon)]$, then he still wins. However, his utility remains unchanged from that obtained under truthful bidding. If the contractor bids $b_n > x_n + \mathbb{E}_\epsilon[V(y_n+\epsilon)]$, then there are two possibilities: In the first scenario, the contractor still wins (i.e., $b_n \leq b_m$ for all m), but his utility does not change from that obtained under truthful bidding. In the second scenario, the contractor does not win (i.e., $b_n > b_m$ for some $m \neq n$) and his utility is 0, which is lower than that obtained under truthful bidding. **Case 2**: $x_n + \alpha\mathbb{E}_\epsilon[V(y_n+\epsilon)] > \min_{m\neq n} b_m$: If contractor n bids truthfully, then his utility is 0. We now argue that, for contractor n, bidding truthfully weakly dominates all other strategies. If the contractor bids $b_n \geq x_n + \mathbb{E}_\epsilon[V(y_n+\epsilon)]$, then he still loses, and gets a zero payoff. If the contractor bids $b_n < x_n + \mathbb{E}_\epsilon[V(y_n+\epsilon)]$, then there are two possibilities: In the first scenario, the contractor still loses (i.e., $b_n > b_m$ for some m), and gets a zero payoff. In the second scenario, the contractor wins (i.e., $b_n \leq b_m$ for all m). However, his expected utility is $\min_{m\neq n} b_m - x_n - \alpha\mathbb{E}_\epsilon[V(y_n+\epsilon)]$, which is negative and hence, lower than that obtained under truthful bidding. The above two cases complete the proof of Lemma 1. ■

Proof of Lemma A.1. From Lemma 1, we know that every contractor n reports $x_n + \alpha w_n$ as his bid in CS_α. For notational convenience, we drop α from the subscript $n(\alpha)$ and simply write it as (n). To obtain the buyer's expected cost under CS_α in closed-form, we consider two cases: (1) $\delta_X \geq \alpha\delta_W$, (2) $\delta_X \leq \alpha\delta_W$.

Under the assumption $\delta_X \geq \alpha\delta_W$, the density of the convolution $X_n + \alpha W_n$ is:

$$f_{X+\alpha W}(z) = \begin{cases} \frac{1}{\delta_X\delta_W}\left[\frac{z-\underline{x}}{\alpha} - \underline{w}\right], & \text{if} \quad z \in [\underline{x}+\alpha\underline{w}, \underline{x}+\alpha\overline{w}], \\ \frac{1}{\delta_X}, & \text{if} \quad z \in [\underline{x}+\alpha\overline{w}, \overline{x}+\alpha\underline{w}], \\ \frac{1}{\delta_X\delta_W}\left[\overline{w} - \frac{z-\overline{x}}{\alpha}\right], & \text{if} \quad z \in [\overline{x}+\alpha\underline{w}, \overline{x}+\alpha\overline{w}]. \end{cases}$$

Under the assumption $\delta_X \leq \alpha\delta_W$, the density of the convolution $X_n + \alpha W_n$ is:

$$f_{X+\alpha W}(z) = \begin{cases} \frac{1}{\delta_X\delta_W}\left[\frac{z-\underline{x}}{\alpha} - \underline{w}\right], & \text{if} \quad z \in [\underline{x}+\alpha\underline{w}, \overline{x}+\alpha\underline{w}], \\ \frac{1}{\alpha\delta_W}, & \text{if} \quad z \in [\overline{x}+\alpha\underline{w}, \underline{x}+\alpha\overline{w}], \\ \frac{1}{\delta_X\delta_W}\left[\overline{w} - \frac{z-\overline{x}}{\alpha}\right], & \text{if} \quad z \in [\underline{x}+\alpha\overline{w}, \overline{x}+\alpha\overline{w}]. \end{cases}$$

We first consider the case when $\delta_X \geq \alpha\delta_W$. To obtain $C(\alpha)$, we establish the following two expressions:

$$\mathbb{E}[X_{(2)} + \alpha W_{(2)}] = \mu_X + \alpha\mu_W + \frac{\delta_X}{6} + \frac{\delta_W^2}{12\delta_X}\alpha^2 - \frac{\delta_W^3}{60\delta_X^2}\alpha^3, \quad \text{and} \tag{C.1}$$

$$\mathbb{E}[W_{(1)}] = \mu_W - \frac{\delta_W^2}{6\delta_X}\alpha + \frac{\delta_W^3}{20\delta_X^2}\alpha^2. \tag{C.2}$$

We begin by proving (C.1). Using the density of the second-lowest order statistic $X_{(2)} + \alpha W_{(2)}$, we get

$$\mathbb{E}[X_{(2)} + \alpha W_{(2)}] = \int_{\underline{x}+\alpha\underline{w}}^{\overline{x}+\alpha\overline{w}} 2\ z\ f_{X+\alpha W}(z)\ F_{X+\alpha W}(z)\ dz.$$

Let I_1, I_2 and I_3 be the integrals of the function $2zf_{X+\alpha W}(z)F_{X+\alpha W}(z)$ over the intervals $z\in[\underline{x}+\alpha\underline{w},\underline{x}+\alpha\overline{w}]$, $z\in[\underline{x}+\alpha\overline{w},\overline{x}+\alpha\underline{w}]$, and $z\in[\overline{x}+\alpha\underline{w},\overline{x}+\alpha\overline{w}]$, respectively. Using standard algebra, we obtain:

$$
\begin{aligned}
I_1 =& \frac{\underline{x}\delta_W^2}{4\delta_X^2}\alpha^2+\frac{\delta_W^2}{4\delta_X^2}\left[\frac{4\overline{w}}{5}+\frac{\underline{w}}{5}\right]\alpha^3,\\
I_2 =& \frac{2}{\delta_X}\left[\frac{\overline{x}^2}{3}-\frac{\underline{x}\overline{x}}{6}-\frac{\underline{x}^2}{6}\right]-\frac{2\delta_W}{\delta_X^2}\left[\frac{\overline{x}^2}{3}-\frac{\underline{x}\overline{x}}{6}-\frac{\underline{x}^2}{6}\right]\alpha+\frac{(\overline{w}+5\underline{w})}{6}\alpha-\frac{(\overline{w}+5\underline{w})\delta_W}{6\delta_X}\alpha^2+\frac{\delta_W^2}{6\delta_X}\alpha^2-\frac{\delta_W^3}{6\delta_X^2}\alpha^3,\\
I_3 =& \frac{\overline{x}\delta_W}{\delta_X}\alpha+\frac{\underline{w}\delta_W}{\delta_X}\alpha^2+\frac{\delta_W^2}{3\delta_X}\alpha^2-\frac{\overline{x}\delta_W^2}{4\delta_X^2}\alpha^2-\frac{\underline{w}\delta_W^2}{4\delta_X^2}\alpha^3-\frac{\delta_W^3}{20\delta_X^2}\alpha^3.
\end{aligned}
$$

Given I_1, I_2 and I_3, using algebraic manipulations, it is straightforward to verify (C.1).

We now establish (C.2). For a fixed y and z, the joint density of having $X_{(1)}+\alpha W_{(1)}=z$ and $W_{(1)}=y$ when there are only two contractors is $2f_X(z-\alpha y)f_W(y)[1-F_{X+\alpha W}(z)]$. Then, we have: $\mathbb{E}[W_{(1)}]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}2\ y\ f_X(z-\alpha y)\ f_W(y)\ [1-F_{X+\alpha W}(z)]\ dy\ dz$. Define $I(z)=\int_{-\infty}^{\infty}2\ y\ f_X(z-\alpha y)\ f_W(y)\ [1-F_{X+\alpha W}(z)]\ dy$. For a fixed $y\in[\underline{w},\overline{w}]$, the density $f_X(z-\alpha y)$ takes the value 0 if $y>(z-\underline{x})/\alpha$, $1/\delta_X$ if $y\in[(z-\overline{x})/\alpha,(z-\underline{x})/\alpha]$, and 0 otherwise. We consider the following cases:

- Case 1: $z\in[\underline{x}+\alpha\underline{w},\underline{x}+\alpha\overline{w}]$:

  In this case, we have:

$$
\begin{aligned}
I(z)&=\int_{-\infty}^{\infty}2\ y\ f_X(z-\alpha y)\ f_W(y)\ [1-F_{X+\alpha W}(z)]\ dy=\frac{[1-F_{X+\alpha W}(z)]}{\delta_X\delta_W}\left[\left(\frac{z-\underline{x}}{\alpha}\right)^2-\underline{w}^2\right];\quad\text{therefore}\\
I_1&=\int_{\underline{x}+\alpha\underline{w}}^{\underline{x}+\alpha\overline{w}}I(z)\ dz=\frac{\alpha}{\delta_X}\left[\frac{\overline{w}^2}{3}+\frac{\underline{w}\overline{w}}{3}-\frac{2\underline{w}^2}{3}\right]-\frac{\alpha^2\delta_W^2}{20\delta_X^2}(2\overline{w}+3\underline{w}).
\end{aligned}
$$

- Case 2: $z\in[\underline{x}+\alpha\overline{w},\overline{x}+\alpha\underline{w}]$:

  In this case, we have:

$$
\begin{aligned}
I(z)&=\int_{-\infty}^{\infty}2\ y\ f_X(z-\alpha y)\ f_W(y)\ [1-F_{X+\alpha W}(z)]\ dy=\frac{[1-F_{X+\alpha W}(z)]}{\delta_X\delta_W}\left[\overline{w}^2-\underline{w}^2\right];\quad\text{therefore}\\
I_2&=\int_{\underline{x}+\alpha\overline{w}}^{\overline{x}+\alpha\underline{w}}I(z)\ dz=\frac{\underline{w}+\overline{w}}{\delta_X}\left[\frac{\delta_X}{2}-\frac{\alpha\delta_W}{2}\right].
\end{aligned}
$$

- Case 3: $z\in[\overline{x}+\alpha\underline{w},\overline{x}+\alpha\overline{w}]$:

  In this case, we have:

$$
\begin{aligned}
I(z)&=\int_{-\infty}^{\infty}2\ y\ f_X(z-\alpha y)\ f_W(y)\ [1-F_{X+\alpha W}(z)]\ dy=\frac{[1-F_{X+\alpha W}(z)]}{\delta_X\delta_W}\left[\overline{w}^2-\left(\frac{z-\overline{x}}{\alpha}\right)^2\right];\quad\text{therefore}\\
I_3&=\int_{\overline{x}+\alpha\underline{w}}^{\overline{x}+\alpha\overline{w}}I(z)dz=\frac{\alpha^2\delta_W^2}{20\delta_X^2}(3\overline{w}+2\underline{w}).
\end{aligned}
$$

Given I_1, I_2 and I_3, using algebraic manipulations, it is straightforward to verify (C.2). By combining (C.1) and (C.2), we obtain the first equation of (A.1).

For $\delta_X\le\alpha\delta_W$, we can follow similar arguments as above and show $\mathbb{E}[X_{(2)}+\alpha W_{(2)}]=\mu_X+\alpha\mu_W+\frac{\delta_W}{6}\alpha+\frac{\delta_X^2}{12\delta_W}\frac{1}{\alpha}-\frac{\delta_X^3}{60\delta_W^2}\frac{1}{\alpha^2}$, and $\mathbb{E}[W_{(1)}]=\mu_W-\frac{\delta_W}{6}+\frac{\delta_X^2}{12\delta_W}\frac{1}{\alpha^2}-\frac{\delta_X^3}{30\delta_W^2}\frac{1}{\alpha^3}$. These expressions result in the second equation of (A.1). This completes the proof of Lemma A.1. ■

Proof of Lemma A.2. We first consider the case when $\delta_X/\delta_W \le 2/9$. Define $\delta_r = \delta_X/\delta_W$. We show that $C(\alpha)$ is quasi-convex in α. In particular, for $\alpha \in [0,\delta_r]$, we demonstrate that $C(\alpha)$ is decreasing in α. When $\alpha \in [\delta_r, 1]$, we demonstrate that $C(\alpha)$ is initially decreasing and later increasing.

We begin with considering $\alpha \in [0,\delta_r]$. In this interval, we have $\delta_X \ge \alpha\delta_W$. Taking the first-order derivative of $C(\alpha)$ in the first equation of (A.1) with respect to α, we obtain: $\frac{\partial}{\partial\alpha}C(\alpha) = -\frac{\delta_W^2}{6\delta_X} + \frac{\delta_W^2}{2\delta_X}\alpha + \frac{\delta_W^3}{10\delta_X^2}\alpha - \frac{\delta_W^3}{5\delta_X^2}\alpha^2$. We note that the roots of this expression are α_1^r and α_2^r, specified as $\alpha_1^r = \left(\frac{1}{4} + \frac{5\delta_X}{4\delta_W}\right) - \sqrt{\left(\frac{1}{4} + \frac{5\delta_X}{4\delta_W}\right)^2 - \frac{5\delta_X}{6\delta_W}}$, and $\alpha_2^r = \left(\frac{1}{4} + \frac{5\delta_X}{4\delta_W}\right) + \sqrt{\left(\frac{1}{4} + \frac{5\delta_X}{4\delta_W}\right)^2 - \frac{5\delta_X}{6\delta_W}}$. Given our case assumption $\delta_X/\delta_W \le 2/9$, it is fairly easy to show that $\alpha_1^r \ge \delta_r$. This, together with the definition of α_1^r and the fact that $\frac{d}{d\alpha}C(\alpha)$ is quadratic and concave, imply that $C(\alpha)$ is decreasing in α over the interval $[0,\delta_r]$.

We now consider the case $\alpha \in [\delta_r, 1]$. In this case, we have $\delta_X \le \alpha\delta_W$. Using the second equation of (A.1), the first-order and second-order derivatives of $C(\alpha)$ with respect to α are: $\frac{d}{d\alpha}C(\alpha) = \frac{\delta_W}{3} - \frac{\delta_X^2}{6\delta_W}\frac{1}{\alpha^3} - \frac{\delta_X^3}{30\delta_W^2}\frac{1}{\alpha^3} + \frac{\delta_X^3}{10\delta_W^2}\frac{1}{\alpha^4}$, and $\frac{d^2}{d\alpha^2}C(\alpha) = \frac{\delta_X^2}{10\delta_W\alpha^5}\left[\left(5 + \frac{\delta_X}{\delta_W}\right)\alpha - \frac{4\delta_X}{\delta_W}\right]$. In the interval $[\delta_r, 1]$, we have $\alpha \ge \delta_r = \delta_X/\delta_W$. This gives: $\frac{d^2}{d\alpha^2}C(\alpha) \ge \frac{\delta_X^2}{10\delta_W\alpha^5}\left[\left(5 + \frac{\delta_X}{\delta_W}\right)\frac{\delta_X}{\delta_W} - \frac{4\delta_X}{\delta_W}\right] = \frac{\delta_X^2}{10\delta_W\alpha^5}\left[\frac{\delta_X}{\delta_W} + \left(\frac{\delta_X}{\delta_W}\right)^2\right] > 0$. Thus, $\frac{d}{d\alpha}C(\alpha)$ is increasing over the interval $\alpha \in [\delta_r, 1]$. We now find its value at the two extremes $\alpha = \delta_r$ and $\alpha = 1$. Given our case assumption $\delta_X/\delta_W \le 2/9$, we find that $\frac{d}{d\alpha}C(\alpha)\Big|_{\alpha=\delta_r} = \frac{9\delta_W}{30}\left(1 - \frac{2}{9}\frac{\delta_W}{\delta_X}\right) \le 0$, and $\frac{d}{d\alpha}C(\alpha)\Big|_{\alpha=1} = \frac{\delta_W}{3}\left[1 - \frac{1}{2}\left(\frac{\delta_X}{\delta_W}\right)^2 + \frac{1}{5}\left(\frac{\delta_X}{\delta_W}\right)^3\right]$ whereby we note that $1 - \frac{1}{2}\left(\frac{\delta_X}{\delta_W}\right)^2 + \frac{1}{5}\left(\frac{\delta_X}{\delta_W}\right)^3$ is a decreasing function of δ_X/δ_W when $\delta_X/\delta_W \in (0, 2/9]$ and its minimum value at $\delta_X/\delta_W = 2/9$ is equal to 0.9725. Thus, $\frac{d}{d\alpha}C(\alpha)$ at $\alpha = 1$ is greater than 0. Given that the slope of $C(\alpha)$ is increasing over the interval $\alpha \in [\delta_r, 1]$, and goes from a negative value at $\alpha = \delta_r$ to a positive value at $\alpha = 1$ indicates that there exists $\alpha \in [\delta_r, 1]$ such that $\frac{d}{d\alpha}C(\alpha) = \frac{\delta_W}{3} - \frac{\delta_X^2}{6\delta_W}\frac{1}{\alpha^3} - \frac{\delta_X^3}{30\delta_W^2}\frac{1}{\alpha^3} + \frac{\delta_X^3}{10\delta_W^2}\frac{1}{\alpha^4} = 0$. Thus, $C(\alpha)$ is quasi-convex and its minimizer α^{opt} is obtained from solving (A.2).

Next, we consider the case when $\delta_X/\delta_W \ge 2/9$. We show that $C(\alpha)$ is quasi-convex in α. In particular, $C(\alpha)$ is decreasing in α for $\alpha \in [0, \alpha_1^r]$, and increasing in α for $\alpha \in [\alpha_1^r, \delta_r]$ and $\alpha \in [\delta_r, 1]$.

We begin by considering the case $\alpha \in [0,\delta_r]$. In this case, we have $\delta_X \ge \alpha\delta_W$. Further, it is easy to see that $\alpha_1^r \le \delta_r \le \alpha_2^r$. This, along with the definition of α_1^r, and the fact that $\frac{d}{d\alpha}C(\alpha)$ is quadratic and concave in α proves that $C(\alpha)$ is decreasing in α over the interval $[0, \alpha_1^r]$, and increasing in α over the interval $[\alpha_1^r, \delta_r]$.

We now consider the case $\alpha \in [\delta_r, 1]$. In this case, we have $\delta_X \le \alpha\delta_W$. The first-order and second-order derivatives of $C(\alpha)$ are: $\frac{d}{d\alpha}C(\alpha) = \frac{\delta_W}{3} - \frac{\delta_X^2}{6\delta_W}\frac{1}{\alpha^3} - \frac{\delta_X^3}{30\delta_W^2}\frac{1}{\alpha^3} + \frac{\delta_X^3}{10\delta_W^2}\frac{1}{\alpha^4}$, and $\frac{d^2}{d\alpha^2}C(\alpha) = \frac{\delta_X^2}{10\delta_W\alpha^5}\left[\left(5 + \frac{\delta_X}{\delta_W}\right)\alpha - \frac{4\delta_X}{\delta_W}\right]$. Given that $\alpha \ge \delta_r$, it is easy to see that $\frac{d^2}{d\alpha^2}C(\alpha) > 0$. Further, $\frac{d}{d\alpha}C(\alpha)\Big|_{\alpha=\delta_r} = \frac{9\delta_W}{30}\left(1 - \frac{2}{9}\frac{\delta_W}{\delta_X}\right)$, which is positive given our case assumption $\delta_X/\delta_W \ge 2/9$. Thus, $C(\alpha)$ is increasing in α for $\alpha \in [\delta_r, 1]$. As a consequence, $C(\alpha)$ is quasi-convex in α and its minimizer $\alpha^{opt} = \alpha_1^r \in [0,\delta_r]$, which is (A.3). This completes our proof of Lemma A.2. ■

Proof of Lemma A.3. Let $X_n' = X_n - \mu_X$ and $W_n' = W_n - \mu_W$ for all n. Let $\gamma = \sqrt{2\sigma_X^2/\alpha^2 + \sigma_W^2}$. Then,

$$\mathbb{E}[W'_{1(\alpha)}] = \mathbb{E}[W_1'; X_1' + \alpha W_1' \le X_2' + \alpha W_2'] + \mathbb{E}[W_2'; X_2' + \alpha W_2' \le X_1' + \alpha W_1'] = 2\mathbb{E}[W_1'; W_1' \le (X_2' - X_1')/\alpha + W_2']$$

$$= 2\int_{-\infty}^{\infty}\left[\int_{-\infty}^{s} w \cdot d(\Phi(w/\sigma_W))\right] d\Phi(s/\gamma) = 2\left[\left(\int_{-\infty}^{s} w \cdot d(\Phi(w/\sigma_W))\right)\Phi(s/\gamma)\right]_{-\infty}^{\infty} - 2\int_{-\infty}^{\infty}\frac{s}{\sigma_W}\phi(s/\sigma_W)\Phi(s/\gamma)ds$$

$$= -2\int_{-\infty}^{\infty}\frac{s}{\sigma_W}\phi(s/\sigma_W)\Phi(s/\gamma)ds = 2\frac{\sigma_W}{\sqrt{2\pi}}\left[\Phi(s/\gamma)e^{-\frac{s^2}{2\sigma_W^2}}\Big|_{-\infty}^{\infty} - \frac{2}{\gamma}\int_{-\infty}^{\infty}\phi(s/\gamma)e^{-\frac{s^2}{2\sigma_W^2}}ds\right]$$

$$= -\frac{\sigma_W}{\sqrt{2\pi}}\frac{2}{\gamma}\int_{-\infty}^{\infty}\phi(s/\gamma)e^{-\frac{s^2}{2\sigma_W^2}}ds = -\frac{\sigma_W^2}{\sqrt{\pi}}\frac{1}{\sqrt{\sigma_X^2/\alpha^2+\sigma_W^2}}$$

Also,

$$\mathbb{E}[X'_{2(\alpha)}+\alpha W'_{2(\alpha)}] = \sqrt{\sigma_X^2+\alpha^2\sigma_W^2}\cdot\mathbb{E}\left[\frac{X'_{2(\alpha)}+\alpha W'_{2(\alpha)}}{\sqrt{\sigma_X^2+\alpha^2\sigma_W^2}}\right] = \sqrt{\sigma_X^2+\alpha^2\sigma_W^2}\frac{1}{\sqrt{\pi}}$$

Thus,

$$C(\alpha) = \mathbb{E}[X_{2(\alpha)}+\alpha W_{2(\alpha)}] + (1-\alpha)\mathbb{E}[W_{1(\alpha)}] = \mathbb{E}[\mu_X + X'_{2(\alpha)} + \alpha\mu_W + \alpha W'_{2(\alpha)}] + (1-\alpha)\mathbb{E}[\mu_W + W'_{1(\alpha)}]$$

$$= \mu_X + \mu_W + \mathbb{E}[X'_{2(\alpha)}+\alpha W'_{2(\alpha)}] + (1-\alpha)\mathbb{E}[W'_{1(\alpha)}] = \mu_X + \mu_W + \frac{\sigma_W}{\sqrt{\pi}}\left[\sqrt{\frac{\sigma_X^2}{\sigma_W^2}+\alpha^2} - \frac{\alpha(1-\alpha)}{\sqrt{\frac{\sigma_X^2}{\sigma_W^2}+\alpha^2}}\right].$$

This completes the proof of Lemma A.3. ■

Proof of Lemma A.4: To characterize the optimal cost-sharing fraction α^{opt}, we first demonstrate that $C(\alpha)$ in Lemma A.3 is quasi-convex in α. Taking the first-order derivative of $C(\alpha)$, we obtain: $\frac{d}{d\alpha}C(\alpha) = (\sigma_X^2\sigma_W^2/\sqrt{\pi})\left((2\alpha^3\sigma_W^2/\sigma_X^2+3\alpha-1)/(\sigma_X^2+\alpha^2\sigma_W^2)^{3/2}\right)$. Note that $2\alpha^3\frac{\sigma_W^2}{\sigma_X^2}+3\alpha-1$ is increasing in α going from a negative value at $\alpha=0$ to a positive value at $\alpha=1$. Thus, $C(\alpha)$ is quasi-convex in α and minimized at α^{opt} satisfying $2\alpha^3\frac{\sigma_W^2}{\sigma_X^2}+3\alpha-1=0$. ■

Proof of Lemma A.5: For $\delta_X \geq \alpha\delta_W$, $C(\alpha)$ is obtained from (A.1). We wish to show that, for a fixed δ_X, the relative gap $(C(\alpha)-OPT_{NC})/OPT_{NC}$ is increasing in $\delta_{W/X}$. We begin by showing that the ratio $[C(\alpha)-OPT_{NC}]/[OPT_{NC}-\mu_X-\mu_W]$ is increasing in $\delta_{W/X}$.

Using the closed-form expressions for $C(\alpha)$ and OPT_{NC}, along with the definition of $\delta_{W/X}$, we obtain:

$$\frac{C(\alpha)-OPT_{NC}}{OPT_{NC}-\mu_X-\mu_W} = \frac{\delta_{W/X}^2\left(-\alpha/6+\alpha^2/4+\alpha^2\delta_{W/X}/20-\alpha^3\delta_{W/X}/15+1/24-\delta_{W/X}/240\right)}{1/6-\delta_{W/X}^2/24+\delta_{W/X}^3/240}$$

It is easy to verify that $1/6-\delta_{W/X}^2/24+\delta_{W/X}^3/240$ is decreasing in $\delta_{W/X}$ for all $\delta_{W/X}\in[0,20/3]$, and is always positive over the interval $\delta_{W/X}\in[0,2]$ (it achieves the value $1/30$ at $\delta_{W/X}=2$). We now show that the numerator $h(\alpha,\delta_{W/X}) = \delta_{W/X}^2\left(-\alpha/6+\alpha^2/4+\alpha^2\delta_{W/X}/20-\alpha^3\delta_{W/X}/15+1/24-\delta_{W/X}/240\right)$ is increasing in $\delta_{W/X}$. This, along with the fact that $h(\alpha,\delta_{W/X})$ is positive, will complete the proof that the ratio $\frac{C(\alpha)-OPT_{NC}}{OPT_{NC}-\mu_X-\mu_W}$ is increasing in $\delta_{W/X}$.

Taking the first-order partial derivative of $h(\alpha,\delta_{W/X})$ with respect to $\delta_{W/X}$, we obtain, $\frac{\partial}{\partial\delta_{W/X}}h(\alpha,\delta_{W/X}) = (12\alpha^2-16\alpha^3-1)\frac{\delta_{W/X}^2}{80}+(1+6\alpha^2-4\alpha)\frac{\delta_{W/X}}{12}$. In what follows, we show that $\frac{\partial}{\partial\delta_{W/X}}h(\alpha,\delta_{W/X})$ is increasing in $\delta_{W/X}$, and its minimum value 0 is attained at $\delta_{W/X}=0$. To get started, we find the second-order and third-order partial derivatives of $h(\alpha,\delta_{W/X})$ with respect to $\delta_{W/X}$ as follows: $\frac{\partial^2}{\partial\delta_{W/X}^2}h(\alpha,\delta_{W/X}) = (12\alpha^2-16\alpha^3-1)\frac{\delta_{W/X}}{40}+(1+6\alpha^2-4\alpha)\frac{1}{12}$, and $\frac{\partial^3}{\partial\delta_{W/X}^3}h(\alpha,\delta_{W/X}) = (12\alpha^2-16\alpha^3-1)\frac{1}{40}$.

The function $12\alpha^2-16\alpha^3-1$ is quasi-concave in α and achieves the maximum value 0 at $\alpha=1/2$. Thus, $\frac{\partial^3}{\partial\delta_{W/X}^3}h(\alpha,\delta_{W/X})\leq 0$ for all $\delta_{W/X}$, which implies that $\frac{\partial}{\partial\delta_{W/X}}h(\alpha,\delta_{W/X})$ is concave in $\delta_{W/X}$. We now check the value of $\frac{\partial^2}{\partial\delta_{W/X}^2}h(\alpha,\delta_{W/X})$ at $\delta_{W/X}=2$. We obtain: $\frac{\partial^2}{\partial\delta_{W/X}^2}h(\alpha,\delta_{W/X})\big|_{\delta_{W/X}=2} = (12\alpha^2-16\alpha^3-1)\frac{1}{20}+(1+6\alpha^2-4\alpha)\frac{1}{12} = -\frac{4}{5}\alpha^3+\frac{11}{10}\alpha^2-\frac{1}{3}\alpha+\frac{1}{30}$, which is initially decreasing in $\alpha\in[0,(11-\sqrt{41})/24]$, then increasing in $\alpha \in [(11-\sqrt{41})/24, (11+\sqrt{41})/24]$, and finally decreasing in $\alpha \in [(11+\sqrt{41})/24, 1]$. Thus, its minimum value is achieved either at $\alpha = (11-\sqrt{41})/24$ or at $\alpha = 1$. Both these values are positive; thus, $\frac{\partial^2}{\partial \delta_{W/X}^2} h(\alpha, \delta_{W/X})$ at $\delta_{W/X} = 2$ is positive. This, along with the concavity of $\frac{\partial}{\partial \delta_{W/X}} h(\alpha, \delta_{W/X})$ imply that $\frac{\partial^2}{\partial \delta_{W/X}^2} h(\alpha, \delta_{W/X})$ is positive for all $\delta_{W/X} \in [0,2]$. Thus, $\frac{\partial}{\partial \delta_{W/X}} h(\alpha, \delta_{W/X})$ is increasing in $\delta_{W/X} \in [0,2]$. Since its minimum value of 0 is attained at $\delta_{W/X} = 0$, it is always positive over the interval $\delta_{W/X} \in [0,2]$. This completes the proof that $\frac{C(\alpha)-OPT_{NC}}{OPT_{NC}-\mu_X-\mu_W}$ is increasing in $\delta_{W/X}$.

Thus, we have $\frac{C(\alpha)-OPT_{NC}}{OPT_{NC}} = h(\alpha, \delta_{W/X}) \left(\frac{1}{6} - \frac{\delta_{W/X}^2}{24} + \frac{\delta_{W/X}^3}{240} + \frac{\mu_X+\mu_W}{\delta_X} \right)^{-1}$, where we use the facts that $\delta_X > 0$ and $1/6 - \delta_{W/X}^2/24 + \delta_{W/X}^3/240 \geq 1/30 > 0$ over the interval $\delta_{W/X} \in [0,2]$. We know that $h(\alpha, \delta_{W/X})$ is increasing in $\delta_{W/X}$, and $1/6 - \delta_{W/X}^2/24 + \delta_{W/X}^3/240$ is decreasing in $\delta_{W/X}$. Thus, for a fixed δ_X, $(C(\alpha) - OPT_{NC})/OPT_{NC}$ is increasing in $\delta_{W/X}$. Also, for a fixed $\delta_{W/X}$, $(C(\alpha) - OPT_{NC})/OPT_{NC}$ is increasing in δ_X. ■

Proof of Lemma A.6: Note that the function $\frac{1}{3\delta_{W/X}} - \frac{\delta_{W/X}}{12} + \frac{\delta_{W/X}^2}{120}$ is decreasing in $\delta_{W/X} \in [0,2]$ and positive. It remains to show that $l(\alpha, \delta_{W/X})$ is increasing in $\delta_{W/X}$ and positive. We check the first-order and second-order derivatives of $l(\alpha, \delta_{W/X})$ with respect to $\delta_{W/X}$ as follows: $\frac{\partial}{\partial \delta_{W/X}} l(\alpha, \delta_{W/X}) = -\frac{\alpha}{3} + \frac{\alpha^2}{2} + \frac{\alpha^2 \delta_{W/X}}{5} - \frac{4\alpha^3 \delta_{W/X}}{15} + \frac{1}{12} - \frac{\delta_{W/X}}{60}$, and $\frac{\partial^2}{\partial \delta_{W/X}^2} l(\alpha, \delta_{W/X}) = \frac{\alpha^2}{5} - \frac{4\alpha^3}{15} - \frac{1}{60}$. It is easy to see that $\frac{\alpha^2}{5} - \frac{4\alpha^3}{15} - \frac{1}{60}$ is quasi-concave in α, and is negative (the maximum value 0 is attained at $\alpha = 1/2$). Thus, $\frac{\partial}{\partial \delta_{W/X}} l(\alpha, \delta_{W/X})$ is decreasing in $\delta_{W/X}$, which gives: $\frac{\partial}{\partial \delta_{W/X}} l(\alpha, \delta_{W/X}) \geq \frac{\partial}{\partial \delta_{W/X}} l(\alpha, \delta_{W/X})\big|_{\delta_{W/X}=2} = -\frac{8}{15}\alpha^3 + \frac{9}{10}\alpha^2 - \frac{1}{3}\alpha + \frac{1}{20}$, which is decreasing in $\alpha \in [0, (9-\sqrt{83/3})/16]$, increasing in $\alpha \in [(9-\sqrt{83/3})/16, (9+\sqrt{83/3})/16]$, and decreasing in $\alpha \in [(9+\sqrt{83/3})/16, 1]$. Its minimum value is achieved at either $\alpha = (9-\sqrt{83/3})/16$ or $\alpha = 1$. Both these values are positive; thus, $\frac{\partial}{\partial \delta_{W/X}} l(\alpha, \delta_{W/X}) \geq 0$ for all $\delta_{W/X} \in [0,2]$. ■

Appendix D: Proof of Theorem 5

We first obtain the optimal cost OPT_e which is later used to establish the near-optimality of the best cost-sharing mechanism. In light of the generalized Revelation Principle (Myerson 1982), we can restrict our search for the optimal procurement mechanism to the class of direct coordination mechanisms represented by a triplet $\{\mathbf{Q}, \mathbf{M}, \mathbf{e}\}$, where $Q_n(\mathbf{y})$ is the probability of selecting contractor n, $M_n(\mathbf{y}, z_n)$ is the payment made to contractor n when the non-cost attribute is z_n, and $e_n(\mathbf{y})$ is the level of effort induced from contractor n. Let $G(e) = ge^2/2$, $V(z) = vz^2/2$, $f_Y(y) = 1/(\overline{y} - \underline{y})$ and $F_Y(y) = (y - \underline{y})/(\overline{y} - \underline{y})$. Then, the buyer's problem can be expressed as follows:

$$OPT_e := \min_{\mathbf{Q},\mathbf{M},\mathbf{e}} \quad \mathbb{E}_{\mathbf{Y}} \left\{ \sum_{n=1}^{N} \left[M_n(\mathbf{Y}, Y_n - e_n(\mathbf{Y})) + Q_n(\mathbf{Y}) V\left(Y_n - e_n(\mathbf{Y})\right) \right] \right\} \tag{D.1}$$

$$\text{s.t.} \quad \mathbb{E}_{\mathbf{Y}_{-n}} \left[M_n(y_n, \mathbf{Y}_{-n}, y_n - e_n(y_n, \mathbf{Y}_{-n})) - Q_n(y_n, \mathbf{Y}_{-n}) G(e_n(y_n, \mathbf{Y}_{-n})) \right] \geq$$

$$\mathbb{E}_{\mathbf{Y}_{-n}} \left[M_n(\hat{y}_n, \mathbf{Y}_{-n}, y_n - \hat{e}_n) - Q_n(\hat{y}_n, \mathbf{Y}_{-n}) G(\hat{e}_n) \right], \quad \forall\ \hat{y}_n, y_n, \hat{e}_n, n, \tag{D.2}$$

$$\mathbb{E}_{\mathbf{Y}_{-n}} \left[M_n(y_n, \mathbf{Y}_{-n}, y_n - e_n(y_n, \mathbf{Y}_{-n})) - Q_n(y_n, \mathbf{Y}_{-n}) G(e_n(y_n, \mathbf{Y}_{-n})) \right] \geq 0, \quad \forall\ y_n, n, \tag{D.3}$$

$$\sum_{n=1}^{N} Q_n(\mathbf{y}) = 1, \quad Q_n(\mathbf{y}) \in \{0, 1\}, \quad \forall\ \mathbf{y}, n, \tag{D.4}$$

where the IC constraints (D.2) ensure that each contractor n reveals his true type y_n and subsequently exerts an effort $e_n(\mathbf{y})$, and the IR constraints (D.3) ensure that each contractor n voluntarily participates in the mechanism.

LEMMA D.1. *The optimal mechanism for problem* (D.1)–(D.4) *is as follows:*

- *The allocation function is*

$$Q_n^\star(\mathbf{y}) = \begin{cases} 1, & \text{if } y_n \le y_m, \forall m, \\ 0, & \text{otherwise.} \end{cases} \tag{D.5}$$

- *The effort exerted by contractor n depends only on his own type y_n (that is, $e_n^\star(\mathbf{y}) = e^\star(y_n)$), and is given by*

$$e^\star(y_n) = \left(\frac{v-g}{v+g}\right) y_n + \left(\frac{g}{v+g}\right) \underline{y}. \tag{D.6}$$

- *Contractor n is paid*

$$M_n^\star(\mathbf{y}, z_n) = G\left(e^\star(y_n)\right) + \int_{y_n}^{\overline{y}} G'\left(e^\star(y)\right) \cdot Q_n^\star\left(y, \mathbf{y}_{-n}\right)\, dy, \tag{D.7}$$

if the realized non-cost attribute $z_n = y_n - e^\star(y_n)$, and is imposed an infinite penalty otherwise.

Proof of Lemma D.1. In problem (D.1)–(D.4), the payment $M_n(y_n', \mathbf{y}_{-n}, y_n - e_n')$ never appears in the objective function (D.1) for any $e_n' \neq y_n - y_n' + e_n(y_n', \mathbf{y}_{-n})$. Thus, we can make the constraint (D.2) most relaxed by setting $M_n(y_n', \mathbf{y}_{-n}, y_n - e_n') = -\infty$ for any $e_n' \neq y_n - y_n' + e_n(y_n', \mathbf{y}_{-n})$. In other words, it suffices to only consider the constraint (D.2) with $e_n' = y_n - y_n' + e_n(y_n', \mathbf{y}_{-n})$. For this reason, the payment $M_n(\mathbf{y}, z_n)$ given to contractor n can be simplified to $\overline{M}_n(\mathbf{y}) = M_n(\mathbf{y}, y_n - e_n(\mathbf{y}))$, where every contractor n finds it beneficial to exert the effort expected by the buyer, given his reported type, to avoid incurring infinite penalty. Let $U_n(\mathbf{y}) = \overline{M}_n(\mathbf{y}) - G(e_n(\mathbf{y}))Q_n(\mathbf{y})$, and $u_n(y_n) = \mathbb{E}_{\mathbf{Y}_{-n}}[U_n(y_n, \mathbf{Y}_{-n})]$ be the utility and expected utility of contractor n, respectively. By replicating the analysis in Laffont and Tirole (1987), we claim that a mechanism $\{\mathbf{Q}, \mathbf{M}, \mathbf{e}\}$ is feasible to (D.1)–(D.4) if and only if all the following conditions hold, for all $\mathbf{y}$ and n:

$$\frac{\partial}{\partial y_n} Q_n(\mathbf{y}) \le 0, \quad \frac{\partial}{\partial y_n} e_n(\mathbf{y}) \le 1, \quad \text{and} \quad \frac{\partial}{\partial y_n} u_n(y_n) = -\mathbb{E}_{\mathbf{Y}_{-n}}\left[G'(e_n(\mathbf{y}))Q_n(\mathbf{y})\right]. \tag{D.8}$$

Let $q_n(y_n) = \mathbb{E}_{\mathbf{Y}_{-n}} Q_n(y_n, \mathbf{Y}_{-n})$. Again, from Laffont and Tirole (1987), we can further show that, under an optimal mechanism, $e_n(\mathbf{y})$ depends only on y_n; thus, without loss of generality, we let $e_n(\mathbf{y}) = e_n(y_n)$. This, along with (D.8), imply

$$u_n(y_n) = u_n(\overline{y}) + \int_{y_n}^{\overline{y}} G'(e_n(y)) \cdot q_n(y)\, dy.$$

Using this, the objective function (D.1) can be transformed to

$$\sum_{n=1}^N u_n(\overline{y}) + \sum_{n=1}^N \mathbb{E}_{\mathbf{Y}} \left\{ Q_n(\mathbf{Y}) \left[G(e_n(Y_n)) + G'(e_n(Y_n)) \frac{F_Y(Y_n)}{f_Y(Y_n)} + V(Y_n - e_n(Y_n)) \right] \right\}, \tag{D.9}$$

which is minimized by setting $u_n(\overline{y}) = 0$ for all n, using $e_n(y_n) = e^\star(y_n)$, where $e^\star(y)$ for any $y \in [\underline{y}, \overline{y}]$ solves

$$\min_{0 \le e \le y} G(e) + G'(e) \frac{F_Y(y)}{f_Y(y)} + V(y - e),$$

and selecting the contractor with the lowest $G(e^\star(y_n)) + G'(e^\star(y_n)) \frac{F_Y(y_n)}{f_Y(y_n)} + V(y_n - e^\star(y_n))$. Given the specific functional forms for $G(\cdot)$ and $V(\cdot)$, it is fairly easy to check that $e^\star(y_n)$ reduces to the expression provided in (D.6) and $G(e^\star(y_n)) + G'(e^\star(y_n)) \frac{F_Y(y_n)}{f_Y(y_n)} + V(y_n - e^\star(y_n))$ is increasing in y_n; thus, the contractor with the lowest y_n is selected as the winner. This completes the proof of Lemma D.1. $\square$

We now use Lemma D.1 to prove Theorem 5. Let $k = g/v$. Using the specific functional forms for $G(\cdot)$ and $V(\cdot)$, the expression for the expected information rent given to the winning contractor under the cost-sharing mechanism CS_α can be simplified to

$$\left[\frac{g}{2}\left(\frac{\alpha}{k+\alpha}\right)^2 + \frac{\alpha v}{2}\left(\frac{k}{k+\alpha}\right)^2\right] \cdot \mathbb{E}[Y_{(2)}^2 - Y_{(1)}^2],$$

which can be easily shown to be increasing in α, after noting that $\mathbb{E}[Y_{(2)}^2 - Y_{(1)}^2] \geq 0$. Similarly, the expected increase in the total cost due to effort inefficiency can be simplified to

$$\left[\frac{g}{2}\left(\frac{\alpha}{k+\alpha}\right)^2 + \frac{v}{2}\left(\frac{k}{k+\alpha}\right)^2\right] \cdot \mathbb{E}[Y_{(1)}^2] - \mathbb{E}\left[G(e^1(Y_{(1)})) + V(Y_{(1)} - e^1(Y_{(1)}))\right],$$

whose first-order derivative with respect to α is $-\frac{vk^2(1-\alpha)}{(k+\alpha)^3}\mathbb{E}[Y_{(1)}^2]$, which is negative for all α. This proves the first part of Theorem 5.

Let $\mu_Y = (\underline{y} + \overline{y})/2$. Under the assumption that Y_n is uniformly distributed, we must have

$$\mathbb{E}[Y_{(1)}] = \underline{y} + \frac{\delta_Y}{N+1}, \quad \text{and} \quad \mathbb{E}[Y_{(1)}^2] = \underline{y}^2 + \underline{y}\frac{2\delta_Y}{N+1} + \frac{2\delta_Y^2}{(N+1)(N+2)}. \tag{D.10}$$

Fix an arbitrary α. Since cost-sharing mechanism is incentive-compatible and individually-rational, it satisfies (D.2)–(D.4). Then, by Laffont and Tirole (1987), it must also satisfy (D.8). Using this, along with the fact that the effort $e_n^\alpha(\cdot)$ depends only on y_n imply that the utility of contractor n under the cost-sharing mechanism, $u_n^\alpha(y_n) = \mathbb{E}_{\mathbf{Y}_{-n}}[M_n^\alpha(y_n, \mathbf{Y}_{-n}, y_n - e^\alpha(y_n)) - G(e^\alpha(y_n))Q_n^\alpha(y_n, \mathbf{Y}_{-n})]$, can be calculated by

$$u_n^\alpha(y_n) := u_n^\alpha(\overline{y}) + \int_{y_n}^{\overline{y}} G'(e^\alpha(y)) \cdot \mathbb{E}_{\mathbf{Y}_{-n}}[Q_n^\alpha(y, \mathbf{Y}_{-n})] \; dy,$$

which allows us to use (D.9) to calculate the expected total cost incurred by the buyer under the cost-sharing mechanism. Here, we note that $u_n^\alpha(\overline{y}) = 0$ for all n under the cost-sharing mechanism. This follows from the fact that if a contractor with the private type $\overline{y}$ wins, he must have tied with another contractor with the same private type, and therefore, his utility is $u_n^\alpha(\overline{y}) = G(e^\alpha(\overline{y})) + \alpha V(\overline{y} - e^\alpha(\overline{y})) - G(e^\alpha(\overline{y})) - \alpha V(\overline{y} - e^\alpha(\overline{y})) = 0$.

Using (D.9), the expected total cost for the buyer under the cost-sharing mechanism is

$$C_e(\alpha) = \mathbb{E}_{Y_{(1)}}\left\{\frac{g}{2}\left(e^\alpha(Y_{(1)})\right)^2 + g \cdot e^\alpha(Y_{(1)})[Y_{(1)} - \underline{y}] + \frac{v}{2}[Y_{(1)} - e^\alpha(Y_{(1)})]^2\right\}.$$

Similarly, the expected total cost for the buyer under the optimal mechanism in Lemma D.1 is

$$OPT_e = \mathbb{E}_{Y_{(1)}}\left\{\frac{g}{2}\left(e^\star(Y_{(1)})\right)^2 + g \cdot e^\star(Y_{(1)})[Y_{(1)} - \underline{y}] + \frac{v}{2}[Y_{(1)} - e^\star(Y_{(1)})]^2\right\}.$$

Direct algebraic manipulation simplifies the expression of OPT_e to

$$OPT_e = g \cdot \left\{\frac{3-k}{2(1+k)}\mathbb{E}[Y_{(1)}^2] - \frac{1-k}{1+k}\; \underline{y}\mathbb{E}[Y_{(1)}] - \frac{k}{2(1+k)}\; \underline{y}^2\right\}. \tag{D.11}$$

Similarly, the expression for $C_e(\alpha)$ is simplified as follows:

$$C_e(\alpha) = g \cdot \left\{\left[\frac{\alpha^2}{2(k+\alpha)^2} + \frac{\alpha}{k+\alpha} + \frac{k}{2(k+\alpha)^2}\right]\mathbb{E}[Y_{(1)}^2] - \frac{\alpha}{k+\alpha}\underline{y}\mathbb{E}[Y_{(1)}]\right\}.$$

Define $t = \frac{\alpha}{k+\alpha}$. Using this definition, we obtain: $C_e(\alpha) = g \cdot \left\{\left[\frac{t^2}{2} + t + \frac{1}{2k}(1-t)^2\right]\mathbb{E}[Y_{(1)}^2] - t\underline{y}\mathbb{E}[Y_{(1)}]\right\}$, which is convex in t. Let t^* denote its minimizer, which is obtained as $t^* = \left(\frac{k}{1+k}\right)\frac{\underline{y}\mathbb{E}[Y_{(1)}]}{\mathbb{E}[Y_{(1)}^2]} + \frac{1-k}{1+k}$. It is easy to check that $t^* < 1$. Correspondingly, $\alpha^{opt} = k/(1-t^*) - k$, which is simplied to: $\alpha^{opt} = \frac{(1-k)\mathbb{E}[Y_{(1)}^2] + \underline{y}k\mathbb{E}[Y_{(1)}]}{2\mathbb{E}[Y_{(1)}^2] - \underline{y}\mathbb{E}[Y_{(1)}]}$.

We now demonstrate the monotonicity of α^{opt} with respect to the problem parameters k, δ_Y, and N. For this, we re-write α^{opt} as:

$$\alpha^{opt} = \frac{1+k}{2 - \underline{y}\mathbb{E}[Y_{(1)}]/\mathbb{E}[Y_{(1)}^2]} - k. \tag{D.12}$$

Given that $\underline{y}\mathbb{E}[Y_{(1)}]/\mathbb{E}[Y_{(1)}^2] < 1$, it is easy to see that α^{opt} is decreasing in k. Further, note that α^{opt} is an increasing function of $\underline{y}\mathbb{E}[Y_{(1)}]/\mathbb{E}[Y_{(1)}^2]$. Using (D.10), it can be shown that

$$\frac{d}{dN}\left(\frac{\underline{y}\mathbb{E}[Y_{(1)}]}{\mathbb{E}[Y_{(1)}^2]}\right) = \frac{1}{(\mathbb{E}[Y_{(1)}])^2} \cdot \frac{\underline{y}\delta_Y}{(N+1)^2} \cdot \left[\underline{y}^2 + 2\underline{y}\delta_Y \frac{(2N+3)}{(N+2)^2} + \frac{2\delta_Y^2}{(N+2)^2}\right],$$

which is positive; thus, α^{opt} is increasing in N. Similarly, it can be shown that

$$\frac{d}{d\delta_Y}\left(\frac{\underline{y}\mathbb{E}[Y_{(1)}]}{\mathbb{E}[Y_{(1)}^2]}\right) = \frac{1}{(\mathbb{E}[Y_{(1)}])^2} \cdot \left[-\frac{\underline{y}^3}{N+1} - \frac{\underline{y}^2\delta_Y(N+10)}{2(N+1)(N+2)} - \frac{2\underline{y}\delta_Y^2}{(N+1)^2} - \frac{\delta_Y^3}{(N+1)^2(N+2)}\right],$$

which is negative; thus, α^{opt} is decreasing in δ_Y. This compeletes the second part of Theorem 5.

We now turn to compute the optimality gap under the best cost-sharing mechanism. Using the expression for α^{opt}, we obtain:

$$C_e(\alpha^{opt}) = g \cdot \left\{\frac{3-k}{2(1+k)}\mathbb{E}[Y_{(1)}^2] - \frac{1-k}{1+k}\,\underline{y}\mathbb{E}[Y_{(1)}] - \frac{k}{2(1+k)}\,\underline{y}^2\frac{(\mathbb{E}[Y_{(1)}])^2}{\mathbb{E}[Y_{(1)}^2]}\right\}. \tag{D.13}$$

Using (D.11) and (D.13), the relative gap in the buyer's expected total cost under the cost-sharing and optimal mechanisms can be obtained as:

$$\frac{C_e(\alpha^{opt}) - OPT_e}{OPT_e} = \frac{1}{OPT_e} \cdot \frac{\underline{y}^2 k}{2(1+k)}\left[\frac{\mathbb{E}[Y_{(1)}^2] - (\mathbb{E}[Y_{(1)}])^2}{\mathbb{E}[Y_{(1)}^2]}\right].$$

We make two observations: (1) If $\underline{y} = 0$, then $C_e(\alpha^{opt}) = OPT_e$. (2) If $N \to \infty$, then $\mathbb{E}[Y_{(1)}] = \underline{y}$ and $\mathbb{E}[Y_{(1)}^2] = \underline{y}^2$ by (D.10); thus, $C_e(\alpha^{opt}) = OPT_e$. In both these cases, the cost-sharing mechanism becomes optimal among all mechanisms. We now consider $\underline{y} > 0$ and a finite N. Again, using (D.10), $\frac{OPT_e}{C_e(\alpha^{opt}) - OPT_e}$ is equal to

$$\begin{aligned}
&\left[\frac{(N+1)^2(N+2)^2 - (12-4k)}{kN(N+2)}\right]\frac{\underline{y}^2}{\delta_Y^2} + \left(\frac{\underline{y}^2}{\delta_Y^2} + \frac{\delta_Y^2}{\underline{y}^2}\right)\left(\frac{12-4k}{kN(N+2)}\right) \\
&+ \left[\frac{6(N+1)(N+2) - (20-4k)}{kN}\right]\frac{\underline{y}}{\delta_Y} + \left(\frac{\underline{y}}{\delta_Y} + \frac{\delta_Y}{\underline{y}}\right)\left(\frac{20-4k}{kN}\right) + \left(\frac{8-2k}{k}\right)(1+1/N) + \frac{8}{k}(1+2/N) \\
> &\left(\frac{\underline{y}^2}{\delta_Y^2} + \frac{\delta_Y^2}{\underline{y}^2}\right)\left(\frac{12-4k}{kN(N+2)}\right) + \left(\frac{\underline{y}}{\delta_Y} + \frac{\delta_Y}{\underline{y}}\right)\left(\frac{20-4k}{kN}\right) + \left(\frac{8-2k}{k}\right)(1+1/N) + \frac{8}{k}(1+2/N) \qquad \text{(D.14)} \\
\geq &2\left(\frac{12-4k}{kN(N+2)}\right) + 2\left(\frac{20-4k}{kN}\right) + \left(\frac{8-2k}{k}\right)(1+1/N) + \frac{8}{k}(1+2/N) \qquad \text{(D.15)} \\
> &\frac{16}{k} - 2 \geq 14, \qquad \text{(D.16)}
\end{aligned}$$

where the inequality (D.14) follows from the facts that $(N+1)(N+2) \geq 6$ and $k > 0$, the inequality (D.15) follows from the facts that $\underline{y} > 0$ and $\delta_Y > 0$, the inequalities in (D.16) follow from the assumption that $k \leq 1$. We obtain $(C_e(\alpha^{opt}) - OPT_e)/OPT_e < 1/14 < 7.2\%$. This completes the third part of Theorem 5. ■