Dynamic Type Matching

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Mar 22, 2016

We consider an intermediary's problem of dynamically matching demand and supply of heterogeneous types in a periodic-review fashion. More specifically, there are two disjoint sets of demand and supply types. There is a reward associated with each possible matching of a demand type and a supply type. In each period, demand and supply of various types arrive in random quantities. The platform's problem is to decide on the optimal matching policy to maximize the total discounted rewards minus costs, given that unmatched demand and supply will incur waiting or holding costs, and will be carried over to the next period with abandonments. For this dynamic matching problem, we provide sufficient conditions (which we call *modified Monge conditions*) only on matching rewards such that the optimal matching policy follows a priority hierarchy among possible matching pairs: if some pair of demand and supply types is not matched as much as possible, all pairs that have strictly lower priority down the hierarchy should not be matched. This result is obtained by a generalization of the classic augmenting path approach and adapt it to the stochastic problem, which can be viewed as a generalized, stochastic and dynamic, assignment/transportation problem.

1. Introduction

Consider a firm that manages the process of matching supply with demand in a periodic-review fashion. There are multiple types of demand and supply, with a reward r_{ij} generated by matching one unit of type i demand and one unit of type j supply. At the beginning of each period, demand and supply of various types arrive in *random* quantities. The firm's problem is to decide how to match them and to what extent, so as to maximize the total discounted rewards minus costs, given that unmatched demand and supply will incur unit waiting and holding cost rates c and h, respectively, and will be carried over to the next period with carry-over rates $\alpha \in [0, 1]$ and $\beta \in [0, 1]$, respectively.

1.1. Motivation

That is exactly the essence of the problem faced by many intermediaries in the sharing economy. Operations management deals with the management of the process of matching supply with demand. There is a new form of such process that calls for active management—a sharing economy with *crowdsourced* supply. For example, carpooling platforms such as iCarpool and UberPool match a driver heading to a destination with several riders to the same destination or in the same direction. Amazon crowdsources inventories of an identical item from third-party merchants to its warehouses, to fulfill online orders.[1] A nonprofit organization, United Network for Organ Sharing (UNOS), allocates donated organs to patients in need of transplantation. These popular business and nonprofit sharing-economy models are based on what academics often call a *two-sided market* (Rochet and Tirole 2006). In such a market, an information technology platform is developed and maintained by an intermediary firm to make sharing-economy activities possible. Three parties are involved, namely, an intermediary firm, the demand side and the supply side. In this structure, the intermediary organization matches demand and supply of *heterogeneous* types.

From the intermediary's perspective, matching different types of demand and supply generates distinct rewards (or equivalently, mismatch costs). That is because, in some cases, types on either side of the market have their own *idiosyncratic* preferences for types on the other side; in other words, types have "taste" differences. For example, consider geographic locations as types in the carpooling example. It is more efficient for the intermediary to dispatch a driver closer to the rider. We say that such a market has *horizontally* differentiated types. In other cases, types on either side of the market have a *uniform* preference for types on the other side; in other words, types have "quality" differences. For example, consider health status as types in organ sharing. A patient's life expectancy after a transplant will be longer if the donated organ or the patient is in a healthier condition. Such a market is said to have *vertically* differentiated types. Even more often, we see a mix of horizontally and vertically differentiated types in a market. For example, in organ sharing, in addition to health status as vertically differentiated types, blood and tissue can be considered as horizontally differentiated types, because rewards are generated only between compatible pairs.

Other than dealing with heterogeneous types, the matching of demand and supply by an intermediary in the sharing economy can be extremely difficult for at least two more reasons. First, there are time-varying uncertainties on both the demand and supply sides, which may be out of the control of the intermediary. Second, arrived but unmatched demand and supply may leave the market over time. Economic theories use the tool of "price" to match demand with supply. While price does play an important role in many marketplaces, especially at the strategic and tactical levels, day-to-day operations often require more than price adjustment to achieve efficiency for practitioners. For example, in ridesharing, though Uber is well-known for its "surge pricing,"

[1] Amazon calls that an "inventory commingling program." A product ordered from Amazon or a third-party seller may not have originated from the original seller. The program gives Amazon the flexibility to ship products on the basis of their geographic proximity to customers, thus shortening delivery times and reducing shipping costs.

the same rate applies to all rides at the same time regardless of their origin and destination; in other words, the rate at any given time is exogenous to geographic locations as "types" of riders and drivers. For another example, the allocation of donated organs in the United States does not involve prices at all. Given that prices are exogenous[2] or irrelevant, intervention at the operational level, by directly matching supply with demand of various types, provides an efficient way for the intermediary organization to allocate the crowdsourced supply across different types of demand. In summary, the intermediary has the task of matching exogenous streams of demand and supply types to maximize total profit or social welfare, taking into account that there will be time-dependent random arrivals of demand and supply in the future and that unmatched demand and supply need to be compensated and may abandon the wait.

1.2. Main Results

We formulate the intermediary firm's dynamic matching problem as a finite-horizon discrete-time stochastic dynamic program and analyze it for the structural properties of optimal matching policies and good heuristic policies. We obtain the following set of *distribution-free* structural results under the assumption that demand and supply distributions have bounded means.

Priority properties. Using only matching rewards, we establish a *modified Monge partial order* that specifies a dominance relation between two pairs of demand and supply types. With this notion of partial order, we provide the following structural properties of an optimal matching policy. First, for *any* two pairs of demand and supply types with one strictly dominating the other, it is optimal to prioritize the matching of the dominating pair over the dominated pair. Second, it is optimal to greedily match a *perfect pair* of demand and supply types that dominates all other pairs sharing its demand or supply type. As theoretic contributions, these results are obtained through generalizing the classic augmenting path approach and adapting it to our stochastic problem through induction.

We proceed to study two cases of the general reward structure: unidirectionally horizontal types and vertical types. For these two reward structures, all neighboring pairs of demand and supply types are shown to be comparable under the partial order. As a result of the general priority properties, the optimal matching policy boils down to a match-down-to structure when considering a specific pair of demand and supply types, along the priority hierarchy. In the optimal policy, if *some* pair of demand and supply types is not matched as much as possible, *all* pairs that are strictly dominated by this pair should not be matched at all.

[2] The pricing part of Uber's practice is at a higher level than the matching part. A higher price can encourage more drivers and discourage more riders to arrive at the market. Given the price is determined and announced, the matching decisions are made at the operational level after drivers and riders see the price and enter the market. Because our structural results are distribution free, they can be useful for the matching decisions in Uber's business practice as well.

Unidirectionally horizontal types. We assume that demand and supply types are located on a line or a circle. The *unidirectional* "distance" between a demand type and a supply type is the distance one travels unidirectionally along the line or circle from the location of the supply type to that of the demand type. The reward for matching two linearly decreases in their "distance." Using the general priority properties, we verify that it is optimal to match as much as possible the two that are closest to each other. Moreover, there exists a priority hierarchy in matching imperfect pairs. For any given demand (or supply) type, the closer its distance to a supply (or demand) type, the higher the priority to match the closer pair.[3] As a result, the optimal matching policy has a *match-down-to* structure: along the priority matching hierarchy, for a pair of demand and supply types, there exist state-dependent thresholds, with those for perfect pairs all equal to zero; if demand and supply levels are higher than the thresholds, they should be matched down to the thresholds; otherwise, they should not be matched.

Vertical types. Each demand or supply type is associated with a quality, and generates a higher reward if matched with a supply or demand type of a higher quality. In particular, we assume the reward of matching a pair is the sum of the contributions brought in by its components, which are increasing in quality. Then the optimal matching policy follows a simple structure, which we call *top-down matching* (in an economic term, assortative mating): line up demand types and supply types in descending order of their "quality" from high to low; match them from the top, down to some level. Thus, the optimal matching policy in any period can be fully determined by a total matching quantity. Moreover, we can take a dynamic perspective on the optimal matching policy: as in the case of horizontal types, in the top-down matching procedure there are match-down-to levels (or equivalently, some protection levels) for any pair of demand and supply types. When demand and supply have the same carry-over rate, we show, by verifying the $L^\natural$-concavity of the value functions of a transformed problem, that the optimal total matching quantity (from the aggregate perspective) or the optimal protection levels (from the dynamic perspective) have monotonicity properties with respect to the system state: An increment in the level of a demand or supply type with higher "quality" leads to a higher optimal matching quantity or lower protection levels.

For a general reward structure, any two pairs of demand and supply types that share a common node may not be comparable. Nevertheless, the priority properties remain to hold for all of those that are indeed comparable. In addition, we provide the following bounds and heuristics for the general problem.

[3] Unfortunately, these results on priority, determined by distances, in general fail to hold if the "distance" is the shortest distance. As a result, one should not optimistically expect a general priority structure to hold for those situations.

Bounds and heuristics. We consider the deterministic counterpart of the stochastic dynamic problem for any period with t amount of remaining time in the horizon and any levels of demand and supply; this can be written as a linear program with $O(n \times m \times t)$ variables. If the shadow prices of its dual problem are known or can be approximated by good proxies (e.g., the typical market prices that would encourage demand and supply to enter the market for a given time period), then the optimal matching decisions of the fluid model for a given period can be obtained or approximated by a linear program with only $n \times m$ variables. We show that the fluid model provides an upper bound on the optimal total surplus of the stochastic model. It is asymptotically optimal to re-solve the linear program for the current time and state and apply the solution as a heuristic policy, when the time and the arrivals of demand and supply are scaled up proportionally.

1.3. Applications and Implications

The priority properties hold for all neighboring pairs in the cases of unidirectionally horizontal types and vertical types. These two cases may seem somewhat restrictive. Nevertheless, they apply to many emerging settings and also include many existing problems as special cases.

In particular, the case of unidirectionally horizontal types has the following applications:

Capacity management with upgrading. Upgrading uses a high-class supply to fulfill a low-class demand, which is widely adopted in the business practice, e.g., in travel industries (see, e.g., Yu et al. 2015) and in production/inventory settings (see, e.g., Bassok et al. 1999). Shumsky and Zhang (2009) study a revenue management problem with fixed initial capacities of various supply types, and demand types can only be upgraded one level higher. Yu et al. (2015) extend this single-step upgrading to allow general upgrading. The reward structure for those upgrading problems is a special case of unidirectionally horizontal types located along a line. Thus our results on unidirectionally horizontal types are applicable to a generalized capacity management problem with (one-step or general) upgrading and random replenishment. The feature of random supply is desirable for upgrading, even for those revenue management settings, not to mention for the production/inventory settings. For example, in car rental, car returns can be random and in airline ticket selling, early cancellations or airplane swaps can result in random capacity changes.

Production-market or product-plant assignment in the long chain. The seminal work of Jordan and Graves (1995), based on a study of the General Motors (GM) production network, point outs that the long chain, as a sparse flexibility configuration, can provide almost as much benefit as full flexibility. Then the long chain has been advocated as a paradigm for production network design. More recently, this topic has been reinvigorated by a stream of fascinating works, e.g., Chou et al. (2010), Simchi-Levi and Wei (2012) and Wang and Zhang (2015), which analytically demonstrate the effectiveness of the long chain. If as in the long chain, each plant is configured to support both a "home" market/product and a "neighboring" market/product, the reward structure for production-market or product-plant assignment is a special case of unidirectional horizontal types located along a circle. Thus our results on unidirectionally horizontal types can apply to a dynamic production-market or product-plant assignment problem with demand and yield uncertainty, given the process flexibility configuration is fixed as the long chain.

Carpooling/load matching along a fixed route. Roadie, an online platform, aims to entice college students and other travelers to earn extra pocket money by delivering large, long-haul items on the way to where they are already going. Platforms such as uShip and Cargomatic feed loads to independent truck drivers along their way. Carpooling platforms such as UberPool match riders heading to the same direction or destination. In those cases, the matching reward has two additive components: The first one is a disutility associated with the distance traveled along the fixed route from the driver's current location to pick up the demand.[4] The second is a utility associated with traveling along the route from the demand's pick-up location to its drop-off location. The former is the unidirectionally horizontal case, whereas the latter is a vertically differentiated attribute, because given the same pick-up location, it is more desirable if the demand's travel distance is longer. We show that if riders and drivers head to the same destination at the end of the route[5], a shorter distance to pick up a rider on the way has a higher priority in matching.

Moreover, the case of vertical types has the following ramifications and applications:

Assortative mating. In an empirical study of the centralized medical residency assignment, Agarwal (2015) assumes a simplified "double-vertical" model in which both the residents and programs have a (preference) utility function that is linear in observable traits of types on the other side. As a result, the matching reward for a pair is in the additive form, as assumed in the case of vertical types. The additive reward form is a special case of supermodular reward functions. For a one-shot setting, a supermodular reward function guarantees assortative mating as a stable matching. That is, high-quality demands are matched with high-quality supplies and low-quality demands with low-quality supplies, and types not matched to each other could not be mated without making at least one of them worse off. Surprisingly, this self-centric assortative mating behavior is also optimal for the centralized planner (Becker 2009, p.114). We show that in a dynamic setting, with the additive reward function, it is optimal for the centralized planner

[4] A little detour can be allowed but tends to be negligible.

[5] In a press release on the launching of UberPool, Uber revealed that "on any given day, the vast majority of uberX trips in NYC have a 'lookalike' trip—a trip that starts near, ends near, and is happening around the same time as another trip"; see `https://newsroom.uber.com/us-new-york/vision-for-the-future-1m-fewer-cars-on-the-road/`.

to perform top-down matching, i.e., assortative mating, up to some level, and save the rest for future. For other reward functions, this optimal structure may break down (see Online Appendix D). This result provides the following insights for a dynamic matching market: First, if the reward function is a general supermodular function other than an additive one, there exist scenarios in which socially efficient matching is not assortative (which seems not revealed before; see Li 2008 for a survey). This phenomenon happens only if the waiting/holding costs are moderate. This is because, in one extreme when the waiting and holding costs are sufficiently high, the intermediary prefers greedy matching within the current period, and assortative mating would emerge, and in the other extreme when the waiting/holding costs are close to zero, the intermediary prefers to hold up the matching until the last period with a bigger pool, and again assortative mating would emerge. Second, it may not be efficient to exhaust all demand and supply types at a given time. As a result, a centralized dating agency, or even a decentralized dating website, may want to limit the number of matching pairs at any time, in anticipation of future arrivals of better-quality men and women.

Inventory management with substitution. Consider that a firm sells a line of vertically differentiated products to multiple demand classes. Customers with their class is known to the firm are flexible with substitution, but is only willing to pay more for a higher-class product or will be compensated for a lower-class product, based on their class. The resulting reward function is in an additive form. Our results on vertical types are readily applicable to this dynamic inventory management setting with substitution and random supply.

Inventory rationing. The literature of inventory rationing considers inter-temporal inventory allocation of a single supply type across multiple demand types (see, e.g., Evans 1968, Veinott 1965, Ha 1997a,b and de Véricourt et al. 2002). Demand from the less valuable types can be rejected given possible future arrivals of demand from the more valuable types. The case of vertical types generalizes the idea of inventory rationing by considering *multiple* supply types (with exogenously random replenishment) and multiple demand types with general abandonment rates, whereas the literature considers *one* supply type (though with endogenized ordering/production decisions) and fully backlogged or lost demand types.

Lastly, other than the above settings in which all neighboring matching pairs are comparable, our results can be applicable to the following settings and imply partial characterization of the optimal matching policy within the set of comparable matching pairs.

Assignment or transportation problem. Consider a decision maker who sends (assigns) supply (resource) to demand (task) and gets a reward for each transshipment (assignment). The static assignment problem dates back to Dantzig (1963). The study of a class of sequential assignment match processes (SAMP), in which resources are waiting to be assigned to a sequential stream of tasks with random attributes, originated with Derman et al. (1972). Those classic assignment problems essentially assume a *stationary* supply side (e.g., assuming a frozen waiting list of patients in need of transplantation). In the literature on dynamic and stochastic network flow problems, Glockner and Nemhauser (2000) formulate the problem as a multistage stochastic linear program and focus on using approximate algorithms to solve the large-scale problem. Powell (1996) and Spivey and Powell (2004) study a dynamic assignment problem that allows random, dynamic arrivals of tasks and resources, and they focus on heuristic policies. For a matching problem with *time-varying* uncertainties on both demand and supply sides that can be viewed as a generalization of the dynamic assignment or transportation problem, we generalize the "augmenting path" approach to derive structural properties of the optimal policy.

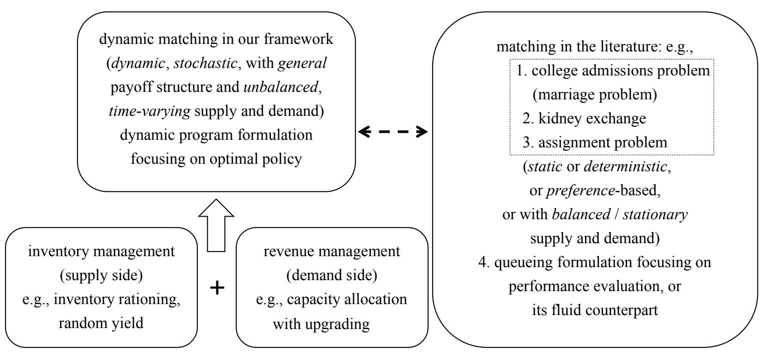
Type mating. A common feature of many manufacturing processes is the mating of two halves to produce a final product. For a flat panel display, the two halves are "active" and "passive" layers of an electronic display. For a ball bearing, the two halves are an inner race and outer race. The location of defects on each half can be examined and the mating of the two that have defects at the same locations generates the highest value. Duenyas et al. (1997) is the first to study this dynamic type mating problem. They assume that *one* unit of a demand and a supply type arrives in each period. Our model generalizes theirs by accounting for arbitrary patterns of demand and supply arrivals and their attrition over time. Other than the partial characterization of the optimal matching policy for those comparable pairs, our result on perfect pairs immediately implies that all two halves that share co-located defects should be matched as much as possible. (Similarly, Uber and Amazon should always match a demand with a supply if they are originated from the same geographic region regardless of future demand and supply uncertainty across all regions.)

2. Literature Review

We illustrate the high-level positioning of our framework with Figure 1. The proposed dynamic-matching framework can be viewed as a generalization of two foundations of operations management, i.e., inventory management where the firm orders the supply centrally (Zipkin 2000), and revenue management where the firm regulates the demand side with a fixed supply side (Talluri and van Ryzin 2006), and of a combination of the two, i.e., joint pricing and inventory control (Chen and Simchi-Levi 2012). Unlike in inventory management and revenue management, the supply in the sharing economy is crowdsourced. It adds complexity beyond existing operations frameworks of stochastic inventory theory and revenue management. With these foundations of operations management, the tools, techniques, and insights developed in those more established areas may be transferred to this new territory. For instance, to derive the monotonicity property of the optimal matching policy, we use $L^\natural$-concavity of the value functions in a transformed system, which has been applied for deriving structural properties for lost-sales inventory models (Zipkin 2008) and for perishable-inventory models (Chen et al. 2014).

More specifically, in connection with inventory management, our framework is closely related to the literature on inventory rationing (see §1.3), and also a stream of research on production systems with random yield. Pioneered by Henig and Gerchak (1990), this stream considers unreliable production that yields only a random portion of the planned quantity. In contrast, our framework considers a class of problems with purely random sources of supply, independently of the firm's decisions; in contrast, the output from a random-yield production system is a random fraction or perturbation of the planned amount. In its connection with revenue management, our framework is closely related to quantity-based revenue management (see, e.g., Talluri and van Ryzin 2006, Part I), in particular, dynamic capacity allocation models with upgrading (see §1.3).

Figure 1 Positioning in the literature.



Driven by real-life applications, economists, computer scientists, and operations researchers have studied a variety of two-sided matching problems (see, e.g., Roth and Sotomayor 1990 for a survey). In particular, these problems include the college admissions problem (with the marriage problem as a special case, see, e.g., Teo et al. 2001), kidney exchange and the online bipartite matching problem. We compare our framework with those problems as follows.

First, the college admissions problem is *preference-based*, with the focus on the stable matchings. It involves parties on the demand and supply sides submitting preferences over options (see, e.g., Ashlagi and Shi 2014). As those matching outcomes such as marriage and college admissions can be life-changing, serious efforts in soliciting preferences are necessary. In contrast, as the sharing economy penetrates into our everyday lives, soliciting preferences may not be practical. For instance, when riders hail a car on Uber, they do not have the option, or may not even bother, to choose a driver to be matched with. It requires the intermediary to associate pairs of demand and supply with rewards, as they arrive, and make matching decisions accordingly. To capture this situation, we assign a "monetary" contribution to a pair of demand and supply types. For example, a lower reward will be assigned if a farther-away car is dispatched. Moreover, the college admissions problem tends to have a *static* or *deterministic* nature. Supply and demand arrive with submitted preferences, before the matching decisions will be made, as in the classical marriage problem. In contrast, our framework, as in inventory and revenue management, emphasizes the *dynamic* and *stochastic* nature of a class of matching problems caused by the growth of the sharing economy and characterized by inter-temporal uncertainties.

Second, similarly to the college admissions problem, patients in kidney exchange have heterogeneous *preferences* over kidneys, subject to blood-type and tissue compatibility. (Note that kidney exchange is different from kidney allocation. In the latter, the organs are harvested from cadaveric donors; see below.) Moreover, in a typical situation the patient and donor arrive in pairs, with an incompatible (or less likely, compatible) patient and donor in each pair. Because of the compatibility issue and the fact that patients and donors arrive in pairs, efficient matching heuristics are focused on cycles, such as two-way exchanges or chains of patient-donor pairs; see, e.g., Roth et al. (2004, 2007). Most relevant to our framework is Ünver (2010), which studies dynamic kidney exchange with inter-temporal random arrivals of patient-donor pairs, and attempts to maximize the number of matched compatible pairs. In contrast, our model allows arbitrary *unbalanced* arrivals of demand and supply, with the objective to maximize social welfare or profit.

Third, online bipartite matching problems have many applications such as allocation of display advertisements. Initiated by Karp et al. (1990), the classic version considers a bipartite graph $G = (U, V, E)$, and assumes that the vertices in U arrive in an "online" fashion. That is, only when a vertex $u \in U$ (e.g., a web viewer) arrives, are its incident edges (e.g., his interests) revealed. Then u can be matched to a previously unmatched adjacent vertex in V (e.g., an advertiser). The objective is to maximize the number of matchings. There are many variants, all with the focus on algorithms' competitive ratios (see Manshadi et al. 2012 for a more recent literature review). The main difference from our model is the "online" feature, other than that there is no clear notation of inventory, with one side (e.g., advertisers) always there and the other (e.g., impressions) getting lost if not matched. Instead of worst-case analysis, we focus on the expected value optimization.

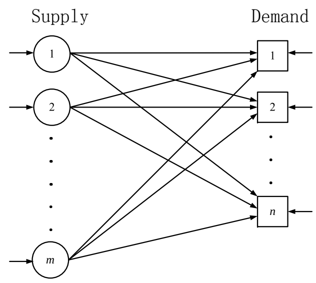
Operations researchers have also been using the queueing approach or its fluid counterpart to study two-sided matching. With a fluid approach of modeling stochastic systems, Zenios et al. (2000) and Su and Zenios (2006) study kidney allocation by exploring the efficiency-equity trade-off, and Akan et al. (2012) study liver allocation by exploring the efficiency-urgency trade-off. Though focusing on structural properties of the optimal policy by exploring the stochastic dynamic program, we also propose a heuristic policy based on a fluid model, and show it is asymptotically optimal. Using double-sided queues, Zenios (1999) studies the transplant waiting list and Afèche et al. (2014) study trading systems like crossing networks. Su and Zenios (2004) analyze a queueing model with service discipline FCFS or LCFS to examine the role of patient choice in the kidney transplant waiting system. Adan and Weiss (2012) show that the stationary distribution of FCFS matching rates for two infinite multi-type sequences is of product form. These papers deal with performance evaluation under a given matching policy. Indeed, Gurvich and Ward (2014) study the dynamic control of matching queues with the objective of minimizing holding costs. The authors observe that in principle, the controller may choose to wait until some "inventory" of items builds up to facilitate more profitable matches in the future. We also make a similar observation.

3. The General Model

We use a boldface letter to denote a vector and its light face with subscript i to denote its i-th entry. By default, a vector is treated as a row vector. We also use a boldface letter to denote a matrix and its light face with subscript ij to denote its (i,j)-th entry. We use $\mathbf{x}_{[k,\ell]}$ to denote the sub-vector of a vector $\mathbf{x}$, containing elements from the k-th entry to the ℓ-th entry. We denote by $\mathbf{e}_\ell^k$ a k-dimensional unit vector where the ℓ-th entry is 1 and all other entries are 0 and by $\mathbf{e}_{ij}^{n\times m}$ an $n\times m$-dimensional matrix where the (i,j)-th entry is 1 and all other entries are 0. We denote by $\mathbf{1}^k$ a k-dimensional vector of 1's and denote by $\mathbf{0}^k$ a k-dimensional vector of 0's (we may omit the superscript k if the dimension of the zero vector is clear from the context). $\mathbb{R}_+=\{r \mid r\geq 0\}$. $x\wedge y=\min\{x,y\}$ and $x\vee y=\max\{x,y\}$. $x^+=\max\{x,0\}$ and $x^-=-\min\{x,0\}$.

Consider a finite horizon with a total number of T periods. In practice, even though demand and supply arrive in continuous time, matching decisions are typically not made in real time. For example, Amazon periodically optimizes the way in which it matches customer orders and its warehouses (see Xu et al. 2009). At the beginning of each period, n types of demand and m types of supply arrive in *random* quantities. Let $\mathcal{D}$ be the set of demand types and $\mathcal{S}$ be the set of supply types. With a slight abuse of notation, we write $\mathcal{D}=\{1,2,\ldots,n\}$ and $\mathcal{S}=\{1,2,\ldots,m\}$, noting that $\mathcal{D}$ and $\mathcal{S}$ are disjoint sets. We use i to index a demand type and j to index a supply type. The pairs of demand and supply are shown in Figure 2 as a bipartite graph. An arc (i,j) represents a match between type i demand and type j supply. For simplicity, we consider a complete bipartite graph in the base model. In other words, any demand type can potentially be matched with any supply type, obviously with different rewards (or equivalently, mismatch costs). (We consider an incomplete bipartite graph in §8.) We denote the complete set of arcs by $\mathcal{A} = \{(i,j) \mid 1 \le i \le n, 1 \le j \le m\}$.

Figure 2 Pairs of demand and supply.



The state for a given period comprises the demand and supply levels of various types before matching but after the arrival of random demand $\mathbf{D} \in \mathbb{R}_+^n$ and supply $\mathbf{S} \in \mathbb{R}_+^m$ for that period. We make *no* assumption about the distributions of random demand and supply of various types except that they have bounded means, i.e., $ED_{it}, ES_{jt} < \infty$ for all i, j and any period t; in other words, our model and its results are *distribution-free*.[6] We denote, as system states, the demand vector by $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$ and the supply vector by $\mathbf{y} = (y_1, \ldots, y_m) \in \mathbb{R}_+^m$, where x_i and y_j are the quantity of type i demand and type j supply available to be matched. Although we assume that the states and the demand and supply arrivals are continuous quantities (and therefore so are the matching decisions), our results can be readily replicated if those quantities are discrete.

On observing the state $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^{n+m}$, the firm decides on the quantity q_{ij} of type i demand to be matched with type j supply, for any $i \in \mathcal{D}$ and $j \in \mathcal{S}$. For conciseness, we write the decision variables of matching quantities in a matrix form as $\mathbf{Q} = (q_{ij}) \in \mathbb{R}_+^{n \times m}$, with $\mathbf{Q}_i$ its i-th row (as a row vector) and $\mathbf{Q}^j$ its j-th column (as a column vector). We assume that there is a reward r_{ij} for matching one unit of type i demand and one unit of type j supply for all i, j. Similarly, we can write the rewards in a matrix form as $\mathbf{R} = (r_{ij}) \in \mathbb{R}^{n \times m}$. Thus the total reward from matching is linear in the matching quantities. That is, $\mathbf{R} \circ \mathbf{Q} \equiv \sum_{i=1}^n \sum_{j=1}^m r_{ij} q_{ij}$, where "$\circ$" gives the sum of elements of the Hadamard product of two matrices. The *post-matching levels* of type i demand and type j supply are given by $u_i = x_i - \mathbf{1}^m \mathbf{Q}_i^\intercal = x_i - \sum_{j'=1}^m q_{ij'}$ and $v_j = y_j - \mathbf{1}^n \mathbf{Q}^j = y_j - \sum_{i'=1}^n q_{i'j}$,

[6] Even if demand and supply distributions have unbounded means, our priority structure is still sample-path-wise optimal.

respectively. That is, $\mathbf{u} = \mathbf{x} - \mathbf{1}^m \mathbf{Q}^\intercal$ and $\mathbf{v} = \mathbf{y} - \mathbf{1}^n \mathbf{Q}$. The post-matching levels cannot be negative; i.e., $\mathbf{u} \geq \mathbf{0}$, $\mathbf{v} \geq \mathbf{0}$.[7]

After the matching is done in each period, each unit of unmatched demand and supply incurs a holding cost c and h respectively. The cost for demand could be loss of goodwill or waiting costs. (We consider type-dependent costs in §8.) Consequently, the total holding cost amounts to $c\mathbf{1}^n \mathbf{u}^\intercal + h\mathbf{1}^m \mathbf{v}^\intercal = c\sum_{i=1}^n u_i + h\sum_{j=1}^m v_j$. The unmatched demand and supply carry over to the next period with carry-over rates α and β, respectively. (We consider type-dependent and random carry-over rates in §8.) In other words, $(1-\alpha)$ fraction of demand and $(1-\beta)$ fraction of supply leave the system. Without loss of generality, we assume they leave the system with zero surplus.

The firm's goal is to determine a matching policy $\mathbf{Q}^* = (q^*_{ij})$ that maximizes the expected total discounted surplus. (Our perspective is the maximizing of social welfare. Alternatively, the formulation can account for profit maximization if r_{ij} is interpreted as the revenue collected from a matching, and c and h are interpreted as the penalty paid to demand and supply for showing up but without a successful match in a period.) Let $V_t(\mathbf{x}, \mathbf{y})$ be the optimal expected total discounted surplus given that it is in period t and the current state is $(\mathbf{x}, \mathbf{y})$. We formulate the finite-horizon problem by using the following stochastic dynamic program:

$$\begin{aligned} V_t(\mathbf{x},\mathbf{y}) &= \max_{\mathbf{Q}\in\{\mathbf{Q}\geq\mathbf{0}|\mathbf{u}\geq\mathbf{0},\mathbf{v}\geq\mathbf{0}\}} H_t(\mathbf{Q},\mathbf{x},\mathbf{y}), \\ H_t(\mathbf{Q},\mathbf{x},\mathbf{y}) &= \mathbf{R}\circ\mathbf{Q} - c\mathbf{1}^n\mathbf{u}^\intercal - h\mathbf{1}^m\mathbf{v}^\intercal + \gamma E V_{t+1}(\alpha\mathbf{u}+\mathbf{D}, \beta\mathbf{v}+\mathbf{S}), \end{aligned} \tag{1}$$

where $\gamma \leq 1$ is the discount factor. The boundary conditions are $V_{T+1}(\mathbf{x},\mathbf{y}) = 0$ for all $(\mathbf{x},\mathbf{y})$, without loss of generality. In other words, at the end of the horizon, all unmatched demand and supply leave the system with zero surplus.

The existence of a solution to the dynamic program (1) is resolved by the following proposition.

PROPOSITION 1. *The functions $H_t(\mathbf{Q},\mathbf{x},\mathbf{y})$ and $V_t(\mathbf{x},\mathbf{y})$ are continuous and concave. Thus, there exists an optimal matching policy $\mathbf{Q}^*_t(\mathbf{x},\mathbf{y})$.*

Though the existence of an optimal policy is guaranteed, in general we expect the *state-dependent* optimal policy to be extremely complex. Next we characterize some of its structural properties.

[7] For simplicity, without formal definitions, we will take the liberty of using consistent notation for the post-matching levels, with its corresponding matching decision. For example, if a matching decision is denoted by $\bar{\mathbf{Q}}$, its corresponding post-matching levels will be denoted by $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$.

4. Priority Properties of the Optimal Policy

One may expect some intuitive properties of the optimal matching policy, e.g., matching a "perfect" pair in some sense, as much as possible. We provide sufficient conditions for such properties. Since we aim to address a general problem that has random dynamics, the conditions would sufficiently guarantee those properties even for a static problem. Therefore, the conditions we will provide are on the reward matrix and independent of any other system parameters. These conditions will guarantee that certain priority structural properties will hold for the dynamic problem at *any* time and with *any* realized demand and supply. For succinctness, we may only present the definitions and results on one side of the market, analogous definitions and results can be easily stated and obtained for the other side by symmetry.

4.1. Modified Monge Partial Order of Arcs

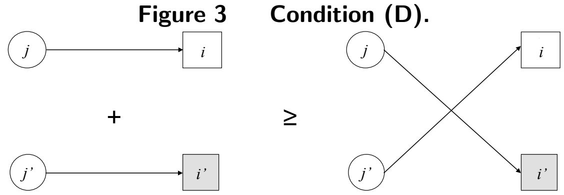
To facilitate discussion, we define a *relation* "$\succeq$" between arcs as follows and will show later it is a *partial order*. First, we consider neighboring arcs in the bipartite graph (Figure 2).

Definition 1 (Modified Monge condition for arcs with a common node). $(i,j) \succeq (i,j')$, if

$$\begin{aligned} &(i) \quad r_{ij} \geq r_{ij'} \quad \text{and} \\ &(ii) \quad r_{ij} + r_{i'j'} \geq r_{ij'} + r_{i'j} \quad \text{for all } i' \in \mathcal{D}. \end{aligned} \tag{D}$$

(When $i' = i$, condition (D) holds automatically. It is easy to see that $(i,j) \succeq (i,j')$ holds automatically for $j' = j$.)

Figure 3 **Condition (D).**



Condition (D) is reminiscent of the *Monge sequence*. Hoffman (1963) provides a necessary and sufficient condition for a transportation problem to be solvable by a greedy algorithm, in which a permutation (called a Monge sequence, and discovered by Gaspard Monge, a French mathematician, in 1781) is followed. A Monge sequence regulates *all* the arcs in the graph, requiring the inequality in condition (D) to hold *only* for all those neighboring arcs (i,j), (i,j') and (i',j) whenever (i,j) precedes (i,j') and (i',j) in the sequence. However, Definition 1 concerns *some* pairs of arcs but requires condition (D) to hold for *all* alternative nodes i' that are different from the common node i. The Monge sequence is introduced to solve a deterministic, demand-supply balanced transportation problem.[8] We propose the partial order, termed as "modified Monge condition," to provide sufficient conditions for structural priority properties in the dynamic demand-supply *unbalanced* matching problem with *random* inter-temporal demand and supply.

Part (i) of Definition 1 requires no less reward by matching pair (i,j) than pair (i,j'). To understand part (ii) of Definition 1, we compare the following two strategies: (1) matching one unit of type i demand and type j supply and another unit of type i' demand and type j' supply, and (2) matching one unit of type i demand and type j' supply and another unit of type i' demand and type j supply. The two strategies have the same post-matching levels of demand and supply. Condition (D) requires that the former strategy weakly dominate the latter (see Figure 3 for an illustration). In other words, part (ii) of Definition 1 implies that there does not exist $i' \in \mathcal{D}$ such that the latter strategy leads to a strictly higher reward than the former. As a result, part (ii) of Definition 1 eliminates the optimality of breaking up the pair (i,j) in matching nodes i, j and j'.

We further define a relation between arcs that do not share any node but can be connected through a sequence of neighboring arcs regulated by the relation "$\succeq$".

Definition 2 (Modified Monge condition for arcs without common nodes). For $i \neq i'$ and $j \neq j'$, we say $(i,j) \succeq (i',j')$ if there exists a sequence of arcs $(i_1,j_1),(i_2,j_2),\ldots,(i_k,j_k)$ such that either $i_k = i_{k+1}$ or $j_k = j_{k+1}$ for $k = 1,\ldots,n-1$, and $(i,j) = (i_1,j_1) \succeq (i_2,j_2) \succeq \cdots \succeq (i_k,j_k) = (i',j')$.

In addition, the *equivalence* relation $(i,j) \simeq (i,j')$ means that $(i,j) \succeq (i,j')$ and $(i,j') \succeq (i,j)$ hold simultaneously. We say $(i,j) \succ (i',j')$ if $(i,j) \succeq (i',j')$ and $(i,j) \not\simeq (i',j')$. We verify below that the relation "$\succeq$" is indeed a partial order (see Online Appendix A for more of its properties).

Lemma 1. *The relation $\succeq$ is a partial order. That is, any arcs ρ_1, ρ_2 and ρ_3 must satisfy: (i) (Reflexivity) $\rho_1 \succeq \rho_1$; (ii) (Antisymmetry) if $\rho_1 \succeq \rho_2$ and $\rho_2 \succeq \rho_1$, then $\rho_1 \simeq \rho_2$; (iii) (Transitivity) If $\rho_1 \succeq \rho_2$ and $\rho_2 \succeq \rho_3$, then $\rho_1 \succeq \rho_3$.*

4.2. Priority Between Two Pairs

We will show if $(i,j) \succeq (i',j')$, (i,j) has a priority over (i',j') in the optimal dynamic matching. To show this, we first need the following lemma, which shows that a matching decision can be weakly improved by transferring quantity from the dominated arc (i',j') to the dominant one (i,j).

Lemma 2. *Suppose $(i,j) \succeq (i',j')$. In period t, if both decisions $\mathbf{Q}$ and $\mathbf{Q} + \epsilon \mathbf{e}_{ij}^{n\times m} - \epsilon \mathbf{e}_{i'j'}^{n\times m}$ are feasible for the state $(\mathbf{x},\mathbf{y})$, then $H_t(\mathbf{Q} + \epsilon \mathbf{e}_{ij}^{n\times m} - \epsilon \mathbf{e}_{i'j'}^{n\times m}, \mathbf{x}, \mathbf{y}) \geq H_t(\mathbf{Q},\mathbf{x},\mathbf{y})$. In other words, the decision $\mathbf{Q} + \epsilon \mathbf{e}_{ij}^{n\times m} - \epsilon \mathbf{e}_{i'j'}^{n\times m}$ weakly dominates $\mathbf{Q}$.*

[8] In the 2-to-2 vertical model of §6, $(2,2),(1,1),(1,2),(2,1)$ is a Monge sequence. But if demand and supply are unbalanced, say $x_1 = 0$, $x_2 = y_1 = y_2 = 1$, matching along the Monge sequence is not optimal.

Sketch of the proof. In period t, for a feasible decision $\mathbf{Q}$, if we transfer ϵ amount from (i',j') to (i,j) (i.e., decrease the matching quantity on (i',j') by ϵ and increase that on (i,j) by ϵ), the immediate benefit for the current period is $\epsilon(r_{ij}-r_{i'j'})\geq 0$. However, as in any dynamic program, this transfer in the current period would also affect the initial states of the next period, hence also affecting all future periods. In particular, after the transfer, the post-matching levels (u_i,v_j) become $(u_i-\epsilon,v_j-\epsilon)$, and $(u_{i'},v_{j'})$ become $(u_{i'}+\epsilon,v_{j'}+\epsilon)$. To decide whether it is profitable to make the transfer now, one needs to evaluate its impact on all future periods as well. Suppose in a future period τ, there exists some type j'' supply that was supposed to be matched with i for an amount of $\tilde{\eta}^{\tau}_{j''}$ along a sample path. But now because there is short of type i demand due to the transfer, one can replace i by i'. Such a replacement has an expected impact of $E(\tilde{\eta}^{\tau}_{j''})(r_{i'j''}-r_{ij''})$ for period τ. The following lemma suggests that the impact on the value functions due to the transfer from (i',j') to (i,j) in period $t-1$ is no worse than the sum of expected impacts due to replacements of i by i' and j by j' for all future periods from period t on. The proof is by induction.

Lemma 3. *In period t, for given $(\mathbf{x},\mathbf{y})$ with $x_i>0$ and $y_j>0$, $\epsilon^1_t\in[0,x_i]$ and $\epsilon^2_t\in[0,y_j]$, there exist $\eta^{\tau}_{j''}\geq 0$ and $\xi^{\tau}_{i''}\geq 0$ for $j''\in\mathcal{S}$, $i''\in\mathcal{D}$ and $\tau=t,\ldots,T+1$ such that $\sum_{\tau=t}^{T}\sum_{j''\in\mathcal{S}}\eta^{\tau}_{j''}\leq\epsilon^1_t$, $\sum_{\tau=t}^{T}\sum_{i''\in\mathcal{D}}\xi^{\tau}_{i''}\leq\epsilon^2_t$ and*

$$V_t(\mathbf{x}-\epsilon^1_t\mathbf{e}^n_i+\epsilon^1_t\mathbf{e}^n_{i'},\mathbf{y}-\epsilon^2_t\mathbf{e}^m_j+\epsilon^2_t\mathbf{e}^m_{j'})-V_t(\mathbf{x},\mathbf{y})\geq\sum_{\tau=t}^{T}\gamma^{\tau-t}[\sum_{j''\in\mathcal{S}}\eta^{\tau}_{j''}(r_{i'j''}-r_{ij''})+\sum_{i''\in\mathcal{D}}\xi^{\tau}_{i''}(r_{i''j'}-r_{i''j})].$$

We further bound the sum of expected future impacts (i.e., the right hand side of the inequality in Lemma 3) to be no worse than $\epsilon(-r_{ij}+r_{i'j'})\leq 0$, which leads to the following result.

Lemma 4. *Suppose $(i,j)\succeq(i',j')$. In period t, for given $(\mathbf{x},\mathbf{y})$, $\epsilon_1\in(0,x_i]$ and $\epsilon_2\in(0,y_j]$, we have $V_t(\mathbf{x}-\epsilon_1\mathbf{e}^n_i+\epsilon_1\mathbf{e}^n_{i'},\mathbf{y}-\epsilon_2\mathbf{e}^m_j+\epsilon_2\mathbf{e}^m_{j'})-V_t(\mathbf{x},\mathbf{y})\geq\epsilon(r_{i'j'}-r_{ij})$, where $\epsilon=\max\{\epsilon_1,\epsilon_2\}$.*

Since the immediate benefit from the transfer is $\epsilon(r_{ij}-r_{i'j'})$ and by Lemma 4, the impact of the transfer on future periods can be bounded by $\epsilon(-r_{ij}+r_{i'j'})$, the overall effect of the transfer is nonnegative. □

Lemma 2 finds an improvement by transferring some matching quantity from the dominated arc (i',j') to the dominant arc (i,j). The result itself is reminiscent of the augmenting path approach to many network flow problems. For example, one can formulate a dynamic but *deterministic* transportation problem as a network flow problem (Bookbinder and Sethi 1980), which then can be solved by an augmenting path approach. However, our matching problem is a dynamic one with *random* future demand and supply and involving the intermediary's optimal control along the sample path. With a certain amount of "flow" transferred from (i,j) to (i',j') in period t, the state in the beginning of period $t+1$ will be changed. This requires matching quantities from period $t+1$ to the end of the horizon to change accordingly to remain feasible along a sample path. The change in period t (the transfer from (i',j') to (i,j)) and possible changes in periods $t+1,\ldots,T$ essentially form an "augmenting cycle," which contains directed arcs $i \to j$ and $j' \to i'$. Given the stochastic and dynamic nature of the problem, it is hard, if not impossible, to write the augmenting cycle in a simple, closed form for every sample path. Through backward induction, the proof of Lemma 2 quantifies the expected impact of possible changes in periods $t+1,\ldots,T$, and shows that the overall effect (together with the transfer of matching quantity in period t) is nonnegative. This approach adopts the idea of "augmenting path" for the stochastic dynamic program.

We need the following definitions to facilitate the presentation of our main result on the priority structure. For any arc $(i,j) \in \mathcal{A}$, we define a set of neighboring arcs that are *strictly* dominated by (i,j): $\mathcal{L}_{ij} \stackrel{\text{def}}{=} \{(i'',j) \mid (i,j) \succ (i'',j)\} \cup \{(i,j'') \mid (i,j) \succ (i,j'')\}$. We also define

$$w_{ij} = w_{ij}(\mathbf{Q},\mathbf{x},\mathbf{y}) \stackrel{\text{def}}{=} \min\{x_i - \sum_{j':(i,j')\notin\mathcal{L}_{ij}} q_{ij'},\ y_j - \sum_{i':(i',j)\notin\mathcal{L}_{ij}} q_{i'j}\}.$$

If $w_{ij}=0$, type i or j is exhausted by the matching over arcs outside the set $\mathcal{L}_{ij}$.

Theorem 1 (Partial order implies priority). *Without loss of generality, assume $\mathbf{x} > \mathbf{0}$ and $\mathbf{y} > \mathbf{0}$ in period t.*[9] *There exists an optimal decision $\mathbf{Q}^*$ such that for any $(i,j) \succ (i',j')$, $\min\left\{w^*_{ij}, q^*_{i'j'}\right\} = 0$, i.e., either $\mathbf{Q}^*$ exhausts type i or j over arcs outside $\mathcal{L}_{ij}$, or $q^*_{i'j'}=0$.*

Proof of Theorem 1. Define $\mathcal{A}_1 = \{(i',j') \in \mathcal{A} \mid \nexists (i,j) \in \mathcal{A}$ such that $(i,j) \succ (i',j')\}$ as the set of undominated arcs. Further define $\mathcal{A}_k = \{(i',j') \in \mathcal{A} \backslash (\bigcup_{l=1}^{k-1} \mathcal{A}_l) \mid \nexists (i,j) \in \mathcal{A} \backslash (\bigcup_{l=1}^{k-1} \mathcal{A}_l)$ such that $(i,j) \succ (i',j')\}$ inductively. Since the total number of arcs is finite, only a finite number of $\mathcal{A}_k$'s are non-empty. Let $K+1 = \min\{k \in \mathbb{N} \mid \mathcal{A}_k = \emptyset\}$. Then $\mathcal{A} = \bigcup_{k=1}^{K} \mathcal{A}_k$.

For the given state $(\mathbf{x},\mathbf{y})$ and a feasible decision $\mathbf{Q}$, we construct another feasible decision that satisfies the desired priority property and weakly dominates $\mathbf{Q}$. Consider the following construction.

Step 0. Let $k \leftarrow 1$ and $\widetilde{\mathcal{A}} \leftarrow \mathcal{A}$.

Step 1. Pick $(i,j) \in \widetilde{\mathcal{A}} \bigcap \mathcal{A}_k$.

Step 2. Find a pair of arcs $(i'',j) \prec (i,j)$ and $(i,j'') \prec (i,j)$ such that $q_{i''j} > 0$ and $q_{ij''} > 0$. Let $\mathbf{Q} \leftarrow \mathbf{Q} - \epsilon \mathbf{e}^{n\times m}_{i''j} - \epsilon \mathbf{e}^{n\times m}_{ij''} + \epsilon \mathbf{e}^{n\times m}_{ij} + \epsilon \mathbf{e}^{n\times m}_{i''j''}$, where $\epsilon = \min\{q_{i''j}, q_{ij''}\}$. Repeat this step until we can no longer find such (i'',j) and (i,j''), at which point either $q_{i''j}=0$ for all $(i'',j) \prec (i,j)$ or $q_{ij''}=0$ for all $(i,j'') \prec (i,j)$.

[9] If $x_i = 0$ (or $y_j = 0$), we can delete demand node i (or supply node j) and all its connected arcs, whose matching quantities are set to be zeros.

Step 3. In the case in which $q_{i''j}=0$ for all $(i'',j)\prec(i,j)$, find $(i,j'')\prec(i,j)$ such that $q_{ij''}>0$ and let $\mathbf{Q}\leftarrow\mathbf{Q}-\delta\mathbf{e}_{ij''}^{n\times m}+\delta\mathbf{e}_{ij}^{n\times m}$, where $\delta=\min\{q_{ij''},v_j\}$. Repeat this until either $v_j=0$ or $q_{ij''}=0$ for all $(i'',j)\preceq(i,j)$.

In the case in which $q_{ij''}=0$ for all $(i,j'')\prec(i,j)$, find $(i'',j)\prec(i,j)$ such that $q_{i''j}>0$. Let $\mathbf{Q}\leftarrow\mathbf{Q}-\theta\mathbf{e}_{i''j}^{n\times m}+\theta\mathbf{e}_{ij}^{n\times m}$, where $\theta=\min\{q_{i''j},u_i\}$. Repeat this until either $u_i=0$ or $q_{i''j}=0$ for all $(i,j'')\prec(i,j)$.

At the end of Step 3, one of the followings is true: (i) $u_iv_j=0$; (ii) $q_{i''j}=q_{ij''}=0$ for all $(i'',j)\prec(i,j)$ and $(i,j'')\prec(i,j)$.

Step 4. Find $(i',j')\prec(i,j)$ such that $q_{i'j'}>0$. Let $\mathbf{Q}\leftarrow\mathbf{Q}-\eta\mathbf{e}_{i'j'}^{n\times m}+\eta\mathbf{e}_{ij}^{n\times m}$, where $\eta=\min\{u_i,v_j,q_{i'j'}\}$. Repeat this until either $u_iv_j=0$ or $q_{i'j'}=0$ for all $(i',j')\prec(i,j)$.

Step 5. Let $\widetilde{A}\leftarrow\widetilde{\mathcal{A}}\setminus\{(i,j)\}$. If $\widetilde{A}\bigcap\mathcal{A}_k=\emptyset$, let $k\leftarrow k+1$. Go to Step 1 if $k\leq K$. Stop if $k>K$.

It is easy to see that each $(i,j)\in\mathcal{A}$ is chosen exactly once in Step 1. At the end of Step 2, suppose without loss of generality, that $q_{i''j}=0$ for all $(i'',j)\prec(i,j)$. Then at the end of Step 3, either $v_j=0$ or $q_{ij''}=q_{i''j}=0$ for all $(i,j''),(i'',j)\in\mathcal{L}_{ij}$. In the former case, $y_j-\sum_{(i'',j)\notin\mathcal{L}_{ij}}q_{i''j}=y_j-\sum_{i''\in\mathcal{D}}q_{i''j}=v_j=0$, implying that $w_{ij}=0$. In the latter case, at the end of Step 4, either $q_{i'j'}=0$ for all $(i',j')\prec(i,j)$ which satisfies the desired property, or $u_iv_j=0$. For the case of $u_iv_j=0$, $w_{ij}=\min\left\{x_i-\sum_{(i,j'')\notin\mathcal{L}_{ij}}q_{ij''},y_j-\sum_{(i'',j)\notin\mathcal{L}_{ij}}q_{i''j}\right\}=\min\{x_i-\sum_{j''\in\mathcal{S}}q_{ij''},y_j-\sum_{i''\in\mathcal{D}}q_{i''j}\}=\min\{u_i,v_j\}=0$, where the second equality is due to $q_{ij''}=q_{i''j}=0$ for all $(i,j''),(i'',j)\in\mathcal{L}_{ij}$. Thus, the desired property will be satisfied by any $(i',j')\prec(i,j)$ for a given (i,j). At the end of the whole construction procedure, the desired property will be satisfied by any pair $(i,j)\succ(i',j')$.

By Lemma 2, the construction procedure keeps weakly improving the matching decision via Steps 2, 3 and 4. Moreover, the procedure stops in a finite number of steps. In the end, we obtain a new feasible decision that satisfies the desired property and weakly dominates the original decision. □

Theorem 1 is analogous to an augmenting path algorithm, with Lemma 2 proven. Overall, our approach can be viewed as a generalization of the augmenting path algorithm for the stochastic and dynamic assignment or transportation problem. The northwest corner rule under the assumption of a Monge sequence can completely solve the deterministic and balanced version of those problems in a greedy fashion. For the stochastic version, we show that only priority property preserves even under the modified Monge conditions, a somewhat stronger set of assumptions than the Monge sequence[10]—even a pair has higher priority in the optimal matching, they are not necessarily

[10] If all arcs are comparable under our partial order along the sequence, then it is a Monge sequence. But we do not require all arcs to be comparable in general.

matched in a greedy fashion; when they are not exhausted, all pairs that have strictly lower priority should not be matched.

The condition $(i,j) \succ (i',j')$ is not necessary for the priority property; see Example 1 below. Nevertheless, it is "necessary" in a *robust* sense against all possible scenarios. In other words, if $(i,j) \succ (i',j')$ fails to hold, one can construct a scenario in which (i',j') has a higher priority over (i,j) in the optimal policy. Hence, the modified Monge conditions are arguably the best on the reward matrix only one can hope to guarantee a general priority structure in the optimal policy.

Example 1. Consider $\mathcal{D} = \{1,2,3\}$ and $\mathcal{S} = \{1,2,3\}$. Let $r_{13} = r_{22} = r_{31} = r_{33} = \epsilon$, $r_{12} = r_{21} = N$, $r_{11} = \frac{3}{2}N$, $r_{23} = r_{32} = 2N$, $c = h = \epsilon$, where ϵ is sufficiently small and N is sufficiently large. In the current period, assume $\mathbf{x} = \mathbf{y} = (1,1,0)$. On the one hand, when there is a high chance of type 3 demand or supply arriving in the next period, it is optimal for the firm to save the unit of type 2 demand and the unit of type 2 supply for the future; i.e., $q^*_{2j'} = q^*_{i'2} = 0$ for all $i' \in \mathcal{D}$, $j' \in \mathcal{S}$. On the other hand, it is optimal to fully match the unit of type 1 demand and the unit of type 1 supply, i.e., $q^*_{11} = 1$. Thus, it is optimal to prioritize matching type 1 demand and type 1 supply over matching type 1 demand and type 2 supply. However, in this example, $r_{11} + r_{22} < r_{12} + r_{21}$, and hence the condition $(1,1) \succ (1,2)$ does not hold. $\square$

4.3. Perfect Pair

Next we provide a sufficient condition for a pair of demand and supply types to be matched in a greedy fashion in preference to all other possible matching options.

Theorem 2 (When greedy matching is optimal). *If $(i,j) \succeq (i,j')$ for all $j' \in \mathcal{S}$ and $(i,j) \succeq (i',j)$ for all $i' \in \mathcal{D}$, then $q^*_{ij} = \min\{x_i, y_j\}$.*

Theorem 2 is *not* a direct consequence of Theorem 1. By directly applying Theorem 1, we can only say that under the conditions in Theorem 2, it is optimal for the firm to prioritize the matching of type i demand and type j supply over any other possibilities. However, it may still be possible that the firm has reserved some type i demand and type j supply without greedily matching them.

The conditions in Theorem 2, i.e., $(i,j) \succeq (i,j')$ for all j' and $(i,j) \succeq (i',j)$ for all i', say that the pair (i,j) dominates all other pairs that share type i demand or type j supply. We say that such a pair forms a *perfect pair* in the eyes of the intermediary firm. Example 1 also serves as a counterexample illustrating that conditions in Theorem 2 are not necessary for greedy matching, though one can say that they are "necessary" in a *robust* sense against all possible scenarios. The dominance relations in Theorem 2 contain two sets of conditions on rewards. The first set, $r_{ij} \geq \max_{i' \in \mathcal{D}, j' \in \mathcal{S}} \{r_{ij'}, r_{i'j}\}$, says that the matching between type i demand and type j supply generates the highest reward among other possible uses of those resources. As a result, type i demand and type j supply are the most favorable for each other from their own perspective. However, they may not form a perfect pair from the intermediary's point of view unless another set of conditions is satisfied. The following example illustrates that the condition $r_{ij} \geq \max_{i' \in \mathcal{D}, j' \in \mathcal{S}} \{r_{ij'}, r_{i'j}\}$ is not enough for the intermediary firm to adopt a greedy match. This is because from a centralized planner's perspective, the components of a most favorable pair for each other may be separately paired with others to generate an overall higher reward. This example emphasizes the importance of the second set of conditions in the modified Monge partial order—i.e., condition (D) holds for all i' with any given j' and for all j' with any given i'—for guaranteeing that a greedy match between type i demand and type j supply will be optimal.

Example 2. The claim in Theorem 2 may fail without the set of condition (D)'s even for a single-period model. To see this, consider a one-period example with $\mathcal{D} = \{1, 2, 3\}$ and $\mathcal{S} = \{1, 2, 3\}$. Suppose that $r_{11} = r_{22} = r_{33} = 2N$, $r_{12} = r_{21} = r_{23} = r_{32} = N + \epsilon$, $r_{13} = r_{31} = \epsilon$, where $N > \epsilon > 0$. Here, $r_{22} \geq \max\{r_{21}, r_{23}, r_{12}, r_{32}\}$, i.e., $(2, 2)$ generates the highest reward. In the current period, assume $\mathbf{x} = (1, 1, 0)$ and $\mathbf{y} = (0, 1, 1)$. If we fully match the type 2 demand with the type 2 supply, then the type 1 demand has to be matched with the type 3 supply given there is only one period, leading to a total reward of $r_{22} + r_{13} = 2N + \epsilon$. Alternatively, if we match the type 2 demand with the type 3 supply and match the type 1 demand with the type 2 supply, the total reward is $r_{23} + r_{12} = 2(N + \epsilon)$, which is higher than $r_{22} + r_{13}$, violating the condition $(2, 2) \succeq (2, 3)$. In this example, we see that although the type 2 demand and type 2 supply are the most favorable for each other in terms of generating the highest reward, they are not a perfect pair in the eyes of the centralized planner. □

As an immediate application of Theorem 2, consider demand and supply types that are specified by their locations in an Euclidean space. The reward of matching supply with demand is a fixed prize minus the disutility proportional to the Euclidean distance between the supply location and the demand location. It is easy to verify that a demand type and a supply type from the *same* location forms a perfect pair, and by Theorem 2, they should be matched as much as possible. To see why they are a perfect pair, we have $r_{ii} + r_{i'j'} \geq r_{ij'} + r_{i'i}$ because $d_{i'j'} \leq d_{ij'} + d_{i'i}$, where d_{ij} is the Euclidean distance between the locations of type i demand and type j supply. The latter inequality is simply the triangle inequality. We summarize this result as follows.

Corollary 1. *In an Euclidean space with horizontally differentiated types as locations, it is optimal to greedily match the demand and supply from the same location.*

Corollary 1 suggests that with geographic locations as types, the intermediary firm such as Uber and Amazon should always match a demand with a supply if they are originated from the same geographic region, or practically speaking, if they are sufficiently close to each other.

So far, we assume no specific reward structure. In the following two sections, we will impose two different, intuitive reward structures in which all neighboring arcs are comparable under the partial order. We will rely on the priority properties that have been shown above, as a starting point, to sharpen the characterization of the optimal matching policy for these two reward structures.

5. Horizontally Differentiated Types

In this section, we first consider the model with demand and supply types that are *horizontally* differentiated in the sense that each type has its own heterogeneous "taste."

5.1. The Directed Line Segment, Directed Circle and Undirected Line Segment

We assume that the n demand and m supply types are distributed on a fixed route C (e.g. a line segment) with a given direction. All the demand types have distinct locations (if that is not true, we can simply treat two demand types sharing the same location as the same type) and so do the supply types. For any two types t_1, t_2, we write $t_1 \to t_2$ to denote that t_1 is located before t_2, along the given direction. We denote by $\vec{d}(t_1, t_2)$ the travel distance from the location of t_1 to that of t_2 along the given direction.

The unit matching reward r_{ij} between type i demand and type j supply is a nonincreasing function of the distance between the two types, which is measured as follows. For $i \in \mathcal{D}$ and $j \in \mathcal{S}$ such that $j \to i$, we define $d_{ij} = \vec{d}(j,i)$. For $i \in \mathcal{D}$ and $j \in \mathcal{S}$ such that $i \to j$, we can consider one of the following definitions:

(i) (Directed line segment) $d_{ij} = N$, where N is an arbitrarily large number;

(ii) (Directed circle) $d_{ij} = |C| - \vec{d}(i,j)$, where $|C|$ is the length of the route C;

(iii) (Undirected line segment) $d_{ij} = \vec{d}(i,j)$.

In case (i), a supply type is not allowed to travel counter to the given direction. In this case, C is a directed line segment, on which $i \in \mathcal{D}$ and $j \in \mathcal{S}$ can be matched with each other if and only if $j \to i$. The product upgrading model has the structure of a directed line segment. Figure 4 illustrates the upgrade model with the restriction that only one-level upgrade is allowed (Shumsky and Zhang 2009). Figure 5 illustrates the model that allows general upgrading (Yu et al. 2015).

In case (ii), a type j supply can travel along the given direction to reach a type i demand if $j \to i$. If $i \to j$, j needs to travel to the end of the route along the given direction, then "reappears" at the origin of the route and continues along the direction to reach i. This is equivalent to the

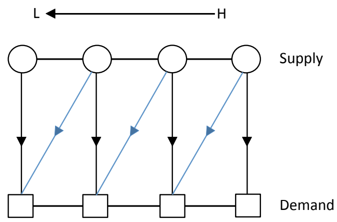


Figure 4 One level upgrade

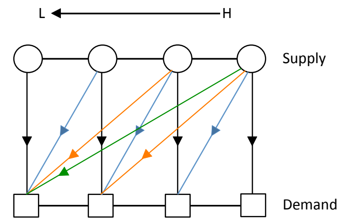


Figure 5 General update

case in which C is a directed (say, clockwise) circle and a supply type always needs to go clockwise on the circle to reach a demand type. The long chain in the process flexibility literature has the structure of a directed circle (with the restriction that a supply type can only be matched with the demand type with the same location or the nearest location along the direction). Figure 6 displays the two equivalent graphic representation of the long chain structure.

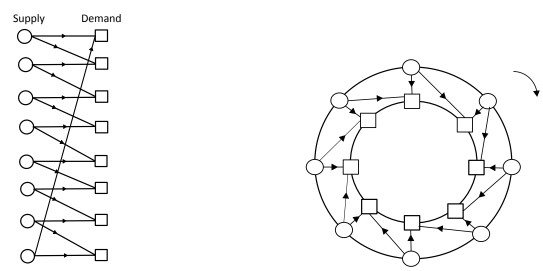


Figure 6 The long chain structure

In case (iii), a supply type can go along or counter to the direction to reach a demand type. Thus the direction no longer plays a role and C is equivalent to an undirected line segment.

As mentioned, the unit matching reward can be written as $r_{ij} = f(d_{ij})$, where f is a nonincreasing function of the distance. If f takes a linear form, we can characterize the priorities in the optimal matching policy. The economic interpretation of a linear function f is that the reward r_{ij} from matching type i demand with type j supply is obtained from a base matching reward r_0 minus the mismatch cost proportional to the distance between type i demand and type j supply.

For $i \in \mathcal{D}$ and $j \in \mathcal{S}$, let $\overleftarrow{(i,j)}$ denote the segment of the route traveled by j to reach i. The following result shows the travel distance can imply the modified Monge partial order, hence implying the matching priority, in the light of Theorem 1.

Theorem 3 (Distance-based priority). *Suppose f is a linear and decreasing function.*

(i) If $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i,j')}$, then $(i,j) \succeq (i,j')$. Similarly, if $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i',j)}$, then $(i,j) \succeq (i',j)$.

(ii) In the case of undirected line segment, if $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i',j')}$ and $\overleftarrow{(i,j)}$ has the same direction with $\overleftarrow{(i',j')}$, then $(i,j) \succeq (i',j')$.

(iii) In the case of directed line segment and circle, $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i',j')}$ *is equivalent to* $(i,j) \succeq (i',j')$*.*

The next result shows that for the directed line segment and the directed circle, each demand or supply type should be matched in a greedy fashion with its closest match.

THEOREM 4 (Greedy match of perfect pairs). *Consider the directed line segment case or the directed circle case. Suppose that* $\overleftarrow{(i,j)}$ *does not contain any other types than themselves. If* f *is nonincreasing and convex,* $q^*_{ij} = \min\{x_i, y_j\}$*.*

Given the analysis in this section so far, there exists an optimal hierarchy for the cases of the directed line segment and circle with linear reward functions. The optimal matching decision in a period can be characterized by state-dependent protection levels $a_{ij}(t,\cdot,\cdot)$ defined in a matching procedure as follows.

To start with, let $k=1$, $(\mathbf{x}^1,\mathbf{y}^1)=(\mathbf{x},\mathbf{y})$ and $\mathbf{Q}^* = \mathbf{0}^{n\times m}$. Also, we represent the set of arcs that have not been matched yet by $\bar{\mathcal{A}}$. Initially, $\bar{\mathcal{A}} = \mathcal{A}$.

Step 1. For each arc $(i,j) \in \left\{(i'',j'') \in \bar{\mathcal{A}} \mid \nexists (i',j') \in \bar{\mathcal{A}} \text{ such that } (i',j') \neq (i'',j'') \text{ and } \overleftarrow{(i',j')} \subseteq \overleftarrow{(i'',j'')}\right\}$ (i.e., (i,j) is undominated in $\bar{\mathcal{A}}$), we do the following.

Step 1.1. Match type i demand with type j supply until their remaining unmatched quantities reach $(x^k_i - y^k_j)^+ + a_{ij}(t,\mathbf{x}^k,\mathbf{y}^k)$ and $(x^k_i - y^k_j)^- + a_{ij}(t,\mathbf{x}^k,\mathbf{y}^k)$ respectively. Remove (i,j) from $\bar{\mathcal{A}}$. Set $q^*_{ij} = x^k_i - (x^k_i - y^k_j)^+ - a_{ij}(t,\mathbf{x}^k,\mathbf{y}^k)$.

Step 1.2. If $a_{ij}(t,\mathbf{x}^k,\mathbf{y}^k) > 0$, then set $q^*_{i'j'} = 0$ and remove (i',j') from $\bar{\mathcal{A}}$ for all $(i',j') \neq (i,j)$ such that $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i',j')}$.

Step 2. Update the state vectors: $\mathbf{x}^{k+1} = \mathbf{x} - \mathbf{1}^m(\mathbf{Q}^*)^\intercal$, $\mathbf{y}^{k+1} = \mathbf{y} - \mathbf{1}^n\mathbf{Q}^*$. Increase k by 1. Go back to Step 1 if $\bar{\mathcal{A}}$ is nonempty, and stop otherwise.

The above procedure performs matching in a priority sequence, where k is the priority level. At a priority level k, the post-matching levels of type i demand and type j supply (right after the matching in Step 1) will be $(x^k_i - y^k_j)^+ + a_{ij}(t,\mathbf{x}^k,\mathbf{y}^k)$ and $(x^k_i - y^k_j)^- + a_{ij}(t,\mathbf{x}^k,\mathbf{y}^k)$ respectively. The level $a_{ij}(t,\mathbf{x}^k,\mathbf{y}^k)$ is the amount we would like to protect from matching. If $a_{ij}(t,\mathbf{x}^k,\mathbf{y}^k) > 0$, all arcs strictly dominated by (i,j) will have zero matching quantities due to the priority structure (see Step 1.2). When $k=1$, each arc (i,j) chosen in Step 1 is undominated by any $(i',j') \in \mathcal{A}$, meaning that type i demand and type j supply will be matched as much as possible. Thus, $a_{ij}(t,\mathbf{x}^1,\mathbf{y}^1) = 0$ for all such (i,j). Another property of $a_{ij}(t,\mathbf{x}^k,\mathbf{y}^k)$ is that it depends on x^k_i and y^k_j only through their difference, $x^k_i - y^k_j$. This is because, if an arc (i,j) is ever selected in Step 1, the decision $\mathbf{Q}^*$

under the state $(\mathbf{x},\mathbf{y})$ will lead to exactly the same post-matching levels as the decision $\mathbf{Q}^*+\epsilon\mathbf{e}_{ij}^{n\times m}$ under the state $(\mathbf{x}+\epsilon\mathbf{e}_i^n,\mathbf{y}+\epsilon\mathbf{e}_j^m)$. Since the total current-period rewards are linear in matching quantities, one can easily verify that $\mathbf{Q}^*+\epsilon\mathbf{e}_{ij}^{n\times m}$ will satisfy the first-order optimality conditions under the state $(\mathbf{x}+\epsilon\mathbf{e}_i^n,\mathbf{y}+\epsilon\mathbf{e}_j^m)$ if $\mathbf{Q}^*$ does so under the state $(\mathbf{x},\mathbf{y})$. Consequently, $\mathbf{Q}^*+\epsilon\mathbf{e}_{ij}^{n\times m}$ is optimal for the state $(\mathbf{x}+\epsilon\mathbf{e}_i^n,\mathbf{y}+\epsilon\mathbf{e}_j^m)$ and has the same protection levels as $\mathbf{Q}^*$.

5.2. Additional Attribute

Consider the reward function $r_{ij}=f(d_{ij})+r_i^a$. The nonincreasing function $f(d_{ij})$ represents the reward associated with traveling by the type j supply to reach the type i demand, and r_i^a represents the reward related to the additional attribute of type i demand, e.g., the reward related to the distance traveled by the pair after their match. The distance traveled after the match usually depends on the need of the customer, and a longer distance suggests a larger reward.

In this case, the same condition in part (i) of Theorem 3, $\overleftarrow{(i,j)}\subseteq\overleftarrow{(i,j')}$, is sufficient for $(i,j)\succeq(i,j')$. Then Theorem 1 would imply that in carpooling, for a given rider, a driver who is closer on the way should have a higher priority to be matched with that rider than another driver who is farther away. In order for $(i,j)\succeq(i',j)$, it suffices to require $f(d_{ij})+r_i^a\geq f(d_{i'j})+r_{i'}^a$ in addition to $\overleftarrow{(i,j)}\subseteq\overleftarrow{(i',j)}$. That is, for a given driver, if a rider is closer and has a longer travel distance, the rider should have a higher priority to be matched, as compared to another rider who is farther and has a shorter travel distance. If all drivers and riders go to the same destination at the end of the fixed route, a driver who picks up a closer-by rider would be better utilized and yield a higher reward. As a result, a shorter unidirectional distance along the fixed route guarantees a higher priority in matching. (A driver who still has capacity after a match can be viewed as new supply.)

Proposition 2. *Suppose that all supply and demand types are located on a directed line segment. Let e be the end point of that line segment, and that $r_i^a=-f(\vec{d}(i,e))$ for all $i\in\mathcal{D}$, where f is a linear and decreasing function. With $r_{ij}=f(d_{ij})+r_i^a$, $(i,j)\succeq(i',j')$ is equivalent to $\overleftarrow{(i,j)}\subseteq\overleftarrow{(i',j')}$.*

5.3. The Model with 2 Supply Types and 2 Demand Types

Next we sharpen the characterization of the optimal matching policy for the model with two demand and supply types, in which demand i has the same location as supply i, for $i=1,2$. Here, the types can be located on the directed line segment, directed circle, or undirected line segment.

Observation 1. For the 2-to-2 horizontal model, $(i,i)\succeq(i,-i)$ and $(i,i)\succeq(-i,i)$ for $i=1,2$.

This is because we have $r_{ii}\geq\max\{r_{i,-i},r_{-i,i}\}$ for $i=1,2$ (as long as f is nonincreasing function) even with d_{ij} defined as the shortest distance along the circle. As a result, by verifying the definition of the modified Monge partial order, it is easy to see that $(1,1)\succeq(1,2)$, $(1,1)\succeq(2,1)$, $(2,2)\succeq(2,1)$

and $(2,2) \succeq (1,2)$ hold. By Theorem 4, type i demand and type i supply, $i=1,2$, should be matched as much as possible. The matching in a period will stop if both supply types or both demand types are exhausted after being matched with their perfect match. Otherwise, it remains to decide the matching quantity between type i demand and type $-i$ supply if they have positive quantities left.

Theorem 5 (2-to-2 horizontal model: optimal matching policy). *Fix an arbitrary period t. For any $(\mathbf{x},\mathbf{y})$, define the type-specific demand and supply imbalance $\eta_i \equiv x_i - y_i$ for $i=1,2$, and the aggregate imbalance $\eta \equiv \eta_1 + \eta_2$. The following matching procedure is optimal: for $i=1,2$,*

(i) Round 1 (Greedy matching for the perfect pair)*: match type i demand and supply as much as possible, i.e., $q^*_{ii} = \min\{x_i, y_i\}$.*

(ii) Round 2 ("Match down to" policy for the imperfect pair)*: if $x_i > y_i$ and $x_{-i} < y_{-i}$, then $q^*_{-i,i} = 0$ and match the imperfect pair of type i demand and type $-i$ supply down to post-matching levels $u^*_i = \eta^+ + a^*_i(t,\eta)$ and $v^*_{-i} = \eta^- + a^*_i(t,\eta)$ respectively, where $a^*_i(t,\eta) = \min\{\bar{a}_i(t,\eta), \eta_i - \eta^+\}$ and $\bar{a}_i(t,\eta)$ is some protection level. Otherwise, $q^*_{i,-i} = 0$.*

Theorem 5 shows the structure of the optimal matching policy for the 2-to-2 horizontal model. In the first round of matching, type i demand is matched as much as possible with its most favorable match, type i supply. After that, if we matched the imperfect pair, type i demand and type $-i$ supply, to the full extent, then the post-matching levels of type i demand and type $-i$ supply would become η^+ and η^- respectively. The optimal matching quantity is characterized by the state-dependent *protection level* $\bar{a}_i(t,\eta)$: the amount $\bar{a}_i(t,\eta)$ is protected from being matched between type i demand and type $-i$ supply so that they are saved for the possible arrival of their perfect match in future periods. The *match-down-to* levels for type i demand and type $-i$ supply after the second round of matching are $\eta^+ + \bar{a}_i(t,\eta)$ and $\eta^- + \bar{a}_i(t,\eta)$ respectively. The matching of the imperfect pair has a match-down-to structure: If the quantity of type i demand, $\eta_i = x_i - y_i$, after the first round of matching is greater than the match-down-to level $\eta^+ + \bar{a}_i(t,\eta)$, then it is optimal to match the imperfect pair and bring the quantity of type i demand down to the level $\eta^+ + \bar{a}_i(t,\eta)$ (and simultaneously, that of type $-i$ supply down to $\eta^- + \bar{a}_i(t,\eta)$). Otherwise, type i demand and type $-i$ supply will not be matched. This structure is analogous to many threshold-type structures in the inventory literature, e.g., the celebrated base-stock policy. Moreover, the match-down-to levels only depend on the *aggregated* discrepancy between total demand and supply across two types. In other words, the match-down-to levels depend on the 4-dimensional state $(\mathbf{x},\mathbf{y})$ *only* through a scalar η. We can obtain a further state collapse in the protection levels for the imperfect matching when the unmatched demand or supply is lost after the matching in each period is done.

Corollary 2 (2-to-2 horizontal model with lost demand or supply). *Suppose that* $x_i > y_i$ *and* $x_{-i} < y_{-i}$. *If* $\alpha = 0$, *there exists a constant* $\hat{v}_{-i}(t)$ *such that the optimal matching quantity between type* i *demand and type* $-i$ *supply is* $q^*_{i,-i} = \eta^-_{-i} - \max\{\hat{v}_{-i}(t) \wedge \eta^-_{-i}, \eta^-\}$. *If* $\beta = 0$, *there exists* $\hat{u}_i(t)$ *such that is* $q^*_{i,-i} = \eta^+_i - \max\{\hat{u}_i(t) \wedge \eta^+_i, \eta^+\}$.

Corollary 2 says that if all the unmatched demand is lost, then the second-round matching reduces the quantity of type $-i$ supply *as close as possible* to a threshold $\hat{v}_{-i}(t)$, which is independent of η. Intuitively, because all the unmatched demand is lost and the post-matching level of supply type i has to be 0 after Round 1 of matching, the intermediary only cares about how much of supply type $-i$ to carry to the next period. This results in a *constant* protection level for supply type $-i$ for the current period. Similarly, if all the unmatched supply is lost, then the optimal matching policy reduces the quantity of type i demand as close as possible to the threshold $\hat{u}_i(t)$.

6. Vertically Differentiated Types

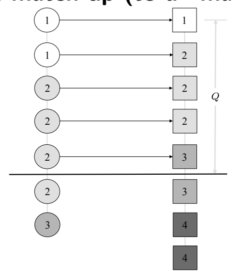
In this section, we consider *vertically* differentiated demand and supply types. Each demand or supply type is associated with a "quality" and generates a higher reward if it is matched with a supply or demand type of a higher quality. Specifically, we consider an additive form of the reward structure: for all $1 \le i \le n$ and $1 \le j \le m$, $r_{ij} = r^d_i + r^s_j$, where r^d_i (or r^s_j) can be understood as the quality of type i demand (or type j supply). Without loss of generality, we index the types such that $r^d_1 > \cdots > r^d_n$ and $r^s_1 > \cdots > r^s_m$. Agarwal (2015) assumes such an additive reward structure.

With the additive reward structure, $r_{ij} + r_{i'j'} = r_{ij'} + r_{i'j}$ for all $i, i' \in \mathcal{D}$ and $j, j' \in \mathcal{S}$. This implies that for two neighboring arcs, $(i,j) \succeq (i',j)$ if and only if $r^d_i \ge r^d_{i'}$, and $(i,j) \succeq (i,j')$ if and only if $r^s_j \ge r^s_{j'}$. This observation can easily be generalized as $(i,j) \succeq (i',j')$ if and only if $i < i'$ and $j < j'$. By Theorem 2, it is optimal to match type 1 demand and type 1 supply in a greedy fashion. From Theorem 1, the arc (i,j) has priority over (i,j') and (i',j) for all $j' > j$ and $i' > i$. This leads to an optimal policy that follows a top-down matching procedure (see Figure 7 for an illustration):

Corollary 3 (Top-down matching). *Line up demand types and supply types separately in increasing order of their indices. Match from the top, down to some level. The optimal matching decision* $\mathbf{Q}$ *in a period is fully determined by a total matching quantity* $Q \stackrel{def}{=} \sum_{i'=1}^n \sum_{j'=1}^m q_{i'j'}$.

Once Q is known, we can recover the matching matrix $\mathbf{Q}$ as follows: Starting with $i = 1$ and $j = 1$, we match type i demand with type j supply until one of them is fully matched or the total matching quantity reaches Q. If type i demand (or type j supply) is fully matched, we increase i (or j) by 1. Then we repeat the above steps until the total matching quantity finally reaches Q.

Figure 7 Line up, match up (to a “match-down-to” level).



For ease of notation we define the following transformed state variables: $\tilde{x}_i \stackrel{\text{def}}{=} \sum_{i'=1}^{i} x_{i'}$ for $1 \le i \le n$, $\tilde{y}_j \stackrel{\text{def}}{=} \sum_{j'=1}^{j} y_{j'}$ for $1 \le i \le m$, and $\tilde{x}_0 \equiv \tilde{y}_0 \equiv 0$. If $\tilde{x}_{i-1} \le Q \le \tilde{x}_i$ and $\tilde{y}_{j-1} \le Q \le \tilde{y}_j$, then types $1, \ldots, i-1$ demand and types $1, \ldots, j-1$ supply are fully matched, an amount $Q - \tilde{x}_{i-1}$ of type i demand is matched with some supply, and an amount $Q - \tilde{y}_{j-1}$ of type j supply is matched with some demand. The rest of the types with quality lower than type i on the demand side and lower than type j on the supply side will not be matched in this period. With the total matching quantity Q, the post-matching levels of demand and supply are given by $u_{i'} = v_{j'} = 0$ for $i' < i$ and $j' < j$, $u_i = \tilde{x}_i - Q$, $v_j = \tilde{y}_j - Q$, $\mathbf{u}_{[i+1,n]} = \mathbf{x}_{[i+1,n]} = (x_{i+1}, \ldots, x_n)$ and $\mathbf{v}_{[j+1,m]} = \mathbf{y}_{[j+1,m]} = (y_{j+1}, \ldots, y_m)$. Then we can rewrite the DP (1) as the following DP with a single decision variable Q:

$$\begin{aligned} V_t(\mathbf{x}, \mathbf{y}) &= \max_{0 \le Q \le \min\{\tilde{x}_n, \tilde{y}_m\}} G_t(Q, \mathbf{x}, \mathbf{y}), \\ G_t(Q, \mathbf{x}, \mathbf{y}) &= \sum_{i'=1}^{n} r_{i'}^d x_{i'} + \sum_{j'=1}^{m} r_{j'}^s y_{j'} - \sum_{i'=1}^{n} \left(r_{i'}^d - r_{i'+1}^d\right)(\tilde{x}_{i'} - Q)^+ - \sum_{j'=1}^{m} \left(r_{j'}^s - r_{j'+1}^s\right)(\tilde{y}_{j'} - Q)^+ \\ &\quad - c(\tilde{x}_n - Q) - h(\tilde{y}_m - Q) + \gamma E V_{t+1}(\alpha \mathbf{u} + \mathbf{D}, \beta \mathbf{v} + \mathbf{S}), \end{aligned} \tag{2}$$

where $r_{n+1}^d = r_{m+1}^s \equiv 0$, $\mathbf{u} = (\mathbf{0}^{i-1}, \tilde{x}_i - Q, \mathbf{x}_{[i+1,n]})$ and $\mathbf{v} = (\mathbf{0}^{j-1}, \tilde{y}_j - Q, \mathbf{y}_{[j+1,m]})$ if $\tilde{x}_{i-1} \le Q < \tilde{x}_i$ and $\tilde{y}_{j-1} \le Q < \tilde{y}_j$. See Online Appendix B for the derivation of DP (2).

Lemma 5. *$G_t(Q, \mathbf{x}, \mathbf{y})$ is concave in Q.*

By Lemma 5, the optimal matching decisions in a period become a one-dimensional convex optimization problem. The following result sharpens the optimal policy characterization. To facilitate the presentation, define $\overline{x}_i = x_i - (\tilde{y}_{j-1} - \tilde{x}_{i-1})^+$ as the available quantity of type i demand before we consider its matching with type j supply, and $\underline{x}_i = (\tilde{x}_i - \tilde{y}_j)^+$ as the remaining quantity of type i demand after we match it with type j supply as much as possible. We define $\overline{y}_j$ and $\underline{y}_j$ similarly.

Theorem 6 (Vertical model: optimal matching procedure). *Consider vertically differentiated types. In the top-down matching procedure, consider matching type i demand with type j supply, which is optimal only if all types $1, \ldots, i-1$ demand and types $1, \ldots, j-1$ supply have been fully matched, and $\tilde{x}_i > \tilde{y}_{j-1}$ and $\tilde{x}_{i-1} < \tilde{y}_j$. There exists a protection level $a_{ij}^*(t)$ depending on*

$(\tilde{x}_i - \tilde{y}_j, \mathbf{x}_{[i+1,n]}, \mathbf{y}_{[j+1,m]})$ such that it is optimal to match type i demand with type j supply until the level of type i demand reduces to $(\tilde{x}_i - \tilde{y}_j)^+ + a^*_{ij}(t)$ if $\overline{x}_i - \underline{x}_i > a^*_{ij}(t)$ (or equivalently, the level of type j supply reduces to $(\tilde{y}_j - \tilde{x}_i)^+ + a^*_{ij}(t)$ if $\overline{y}_j - \underline{y}_j > a^*_{ij}(t)$), and otherwise not to match type i demand with type j supply.

When we come to the decision on matching type i demand with type j supply in the top-down procedure, how much to match is determined by the optimal protection level $a^*_{ij}(t)$. If it is nonzero, the matching procedure would terminate after the matching of type i demand with type j supply; all lower quality types of demand and supply would not be matched. One managerial insight is that higher types tend to be matched in the current period to realize higher immediate reward and lower types with lower "overstocking" costs tend to be saved as safety stock for the future.

Next we consider 3 special cases of the vertical model for which we obtain more structural results.

6.1. Equal Carry-Over Rates

We now consider the case in which demand and supply have the *same* carry-over rate, i.e., $\alpha = \beta$, for which we can further demonstrate monotonicity properties of the optimal matching policy with respect to the system state. To proceed, we define $\tilde{D}_i \stackrel{\text{def}}{=} \sum_{i'=1}^{i} D_{i'}$ and $\tilde{S}_j \stackrel{\text{def}}{=} \sum_{j'=1}^{j} S_{j'}$ for $1 \le i \le n$ and $1 \le j \le m$. We define $\mathbf{U}_k$ as the $k \times k$ upper triangular matrix with all the entries on or above the diagonal equal to one. Then the state transformation can be written in a matrix form: $\mathbf{x}\mathbf{U}_n = \tilde{\mathbf{x}}$ and $\mathbf{y}\mathbf{U}_m = \tilde{\mathbf{y}}$. Also, let $\tilde{V}_t(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \stackrel{\text{def}}{=} V_t(\tilde{\mathbf{x}}\mathbf{U}_n^{-1}, \tilde{\mathbf{y}}\mathbf{U}_m^{-1}) - \tilde{\mathbf{x}}\mathbf{U}_n^{-1}(\mathbf{r}^d)^\intercal - \tilde{\mathbf{y}}\mathbf{U}_m^{-1}(\mathbf{r}^s)^\intercal$.

Since $\mathbf{U}_k^{-1}$ is a $k \times k$ upper-triangular difference matrix that has all diagonal entries equal to 1, $(l, l+1)$-th entry equal to -1 for all $l = 1, 2, k-1$ and all other entries equal to 0, we can rewrite the DP (2) in terms of the value functions $\tilde{V}_t$ and the state variables $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ for $t = 1, \ldots, T$:

$$\begin{aligned}
\tilde{V}_t(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) &= \max_{0 \le Q \le \min\{\tilde{x}_n, \tilde{y}_m\}} \tilde{G}_t(Q, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}), \\
\tilde{G}_t(Q, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) &= -(1-\gamma\alpha)\sum_{i'=1}^{n}(r^d_{i'} - r^d_{i'+1})(\tilde{x}_{i'} - Q)^+ - (1-\gamma\alpha)\sum_{j'=1}^{m}(r^s_{j'} - r^s_{j'+1})(\tilde{y}_{j'} - Q)^+ - c(\tilde{x}_n - Q) \\
&\quad - h(\tilde{y}_m - Q) + \gamma\tilde{\mathbf{D}}\mathbf{U}_n^{-1}(\mathbf{r}^d)^\intercal + \gamma\tilde{\mathbf{S}}\mathbf{U}_m^{-1}(\mathbf{r}^s)^\intercal + \gamma E\tilde{V}_{t+1}(\alpha(\tilde{\mathbf{x}} - Q\mathbf{1}^n)^+ + \tilde{\mathbf{D}}, \alpha(\tilde{\mathbf{y}} - Q\mathbf{1}^m)^+ + \tilde{\mathbf{S}}).
\end{aligned} \tag{3}$$

LEMMA 6. *For all t, $\tilde{V}_t(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ is decreasing in $\tilde{x}_k$ for $1 \le k < n$ and in $\tilde{y}_k$ for $1 \le k < m$.*

To proceed further on the monotonicity properties of the optimal matching policy, we make use of the notion of $L^\natural$-concavity. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called $L^\natural$-convex if $f(\mathbf{x} - \xi\mathbf{1}^n)$ is submodular in $(\mathbf{x}, \xi)$ (see Murota 2003). A function g is $L^\natural$-concave if $-g$ is $L^\natural$-convex.

LEMMA 7. *Suppose $\alpha = \beta$. $\tilde{V}_t(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ is $L^\natural$-concave in $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ for $t = 1, \ldots, T+1$, and $\tilde{G}_t(Q, \tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ is $L^\natural$-concave in $(Q, \tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ for $t = 1, \ldots, T$.*

With the value functions in the transformed system prove to have $L^\natural$-concavity, we obtain the following monotonicity properties of the optimal matching policy for the original system.

Theorem 7 (Vertical model: monotonicity property of optimal total matching quantity). *Suppose $\alpha = \beta$. The optimal total matching quantity $Q_t^*(\mathbf{x},\mathbf{y})$ is nondecreasing in $(\mathbf{x},\mathbf{y})$ and satisfies the following: for $\epsilon > 0$,*

(i) $Q_t^*(\mathbf{x}+\epsilon\mathbf{e}_1^n, \mathbf{y}+\epsilon\mathbf{e}_1^m) = Q_t^*(\mathbf{x},\mathbf{y}) + \epsilon$,

(ii) $0 \le Q_t^*(\mathbf{x}+\epsilon\mathbf{e}_n^n, \mathbf{y}) - Q_t^*(\mathbf{x},\mathbf{y}) \le Q_t^*(\mathbf{x}+\epsilon\mathbf{e}_{n-1}^n, \mathbf{y}) - Q_t^*(\mathbf{x},\mathbf{y}) \le \cdots \le Q_t^*(\mathbf{x}+\epsilon\mathbf{e}_1^n, \mathbf{y}) - Q_t^*(\mathbf{x},\mathbf{y}) \le \epsilon$,

(iii) $0 \le Q_t^*(\mathbf{x}, \mathbf{y}+\epsilon\mathbf{e}_m^m) - Q_t^*(\mathbf{x},\mathbf{y}) \le Q_t^*(\mathbf{x}, \mathbf{y}+\epsilon\mathbf{e}_{m-1}^m) - Q_t^*(\mathbf{x},\mathbf{y}) \le \cdots \le Q_t^*(\mathbf{x}, \mathbf{y}+\epsilon\mathbf{e}_1^m) - Q_t^*(\mathbf{x},\mathbf{y}) \le \epsilon$.

Theorem 7 provides a set of first-order monotonicity properties of the optimal total matching quantity with respect to the state for vertically differentiated types. First, the higher the levels of demand and supply, the more quantities are optimally matched in a period. Second, part (i) is a direct consequence of Theorem 2. It says if the levels of type 1 demand and supply are increased by the same amount, this increased amount will be optimally matched between them in the current period. Third, the series of inequalities (i.e., in parts (ii) and (iii)) show that an increment in the level of a demand or supply type with higher "quality" leads to a higher optimal matching quantity, and the rate of increase is dominated by 1. The statement is consistent with the intuition that higher types are more likely to be matched in the current period. We caution that these results are obtained under the assumption of equal carry-over rates; i.e., $\alpha = \beta$. This is because these monotonicity properties are built upon the $L^\natural$-concavity of the value functions in the transformed system. Unlike concavity and supermodularity, $L^\natural$-concavity depends on the scaling of the variables (Zipkin 2008). (One may expect similar properties to hold for unequal carry-over rates, which may call for a novel form of concavity. We leave that to future research.)

The following corollary recounts Theorem 7 in terms of the state-dependent protection levels.

Corollary 4 (Vertical model: monotonicity property of optimal protection level). *Suppose $\alpha = \beta$. The state-dependent protection level $a_{ij}^*(t, \tilde{x}_i - \tilde{y}_j, \mathbf{x}_{[i+1,n]}, \mathbf{y}_{[j+1,m]})$ is nonincreasing in $(\tilde{x}_i - \tilde{y}_j)^+$, $(\tilde{x}_i - \tilde{y}_j)^-$, $\mathbf{x}_{[i+1,n]}$ and $\mathbf{y}_{[j+1,m]}$, with the decreasing rates no more than 1. In particular, $a_{11}^*(t) \equiv 0$. Moreover, $a_{ij}^*(t)$ is most sensitive to $\tilde{x}_i - \tilde{y}_j$ and is more sensitive to $x_{i'}$ than to $x_{i'+1}$ and to $y_{j'}$ than to $y_{j'+1}$ for $i+1 \le i' \le n-1$ and $j+1 \le j' \le m-1$.*

Lastly, one can consider the one-step-ahead heuristic, in which the intermediary simply optimizes the protection levels for the current period t, in anticipation of implementing greedy matching (i.e., without reserving any demand or supply) from period $t+1$ to the end of the horizon. With $\alpha = \beta$, we can show that the heuristic has a much more simplified state-dependent structure and would be much easier to be computed than the optimal matching policy. See Online Appendix C.

6.2. Lost Demand or Supply

When $\beta = 0$, any unmatched supply does not carry over to the next period. Similarly, $\alpha = 0$ means that unmatched demand will be lost. By symmetry, we focus on the case in which $\beta = 0$.

PROPOSITION 3 (Vertical model: lost supply). *With a stronger assumption $\beta = 0$, Theorem 6 can be strengthened as follows: In considering the matching of type i demand with type j supply, there exists a state-dependent threshold $\theta_{ij}(t, \mathbf{x}_{[i+1,n]})$ such that it is optimal to reduce type i demand to $\theta_{ij}(t, \mathbf{x}_{[i+1,n]})$ if $\tilde{x}_i - \tilde{y}_j < \theta_{ij}(t, \mathbf{x}_{[i+1,n]}) < \tilde{x}_i - \tilde{y}_{j-1}$, to match it with type j supply down to the level $\tilde{x}_i - \tilde{y}_j$ if $\tilde{x}_i - \tilde{y}_j \geq \theta_{ij}(t, \mathbf{x}_{[i+1,n]})$ and otherwise not to match type i demand and type j supply.*

Due to lost supply, the threshold $\theta_{ij}(t, \mathbf{x}_{[i+1,n]})$ that determines how much to match between type i demand and type j supply has a lower-dimensional state dependency, only depending on the time and states of all demand types of lower quality than the focal type i.

6.3. 1 Demand Type and m Supply Types

We consider the model with only 1 demand type and m supply types. The next result immediately follows from Theorem 6.

COROLLARY 5 (1-to-m vertical model). *With a stronger assumption of 1 demand type and m supply types, Theorem 6 can be strengthened as follows: In considering the matching type 1 demand with type j supply, there exists a threshold $\bar{z}_j(t, x_1 - \tilde{y}_j, \mathbf{y}_{[j+1,m]})$ such that it is optimal to reduce the type 1 demand to $\min\{\bar{z}_j(t, x_1 - \tilde{y}_j, \mathbf{y}_{[j+1,m]}), x_1 - \tilde{y}_{j-1}\}$ by matching it with type j supply.*

In the vertical 1-to-m model, the single demand type is matched with supply types sequentially from high quality to low quality. In considering its matching with type j supply, the remaining demand level is $x_1 - \tilde{y}_{j-1}$. There exists an optimal match-down-to level $\bar{z}_j(t, x_1 - \tilde{y}_j, \mathbf{y}_{[j+1,m]})$ such that it is optimal to match the demand down to that level if the available demand is more than the level, and otherwise not to match type 1 demand and type j supply as well as all the lower-quality supply types. Furthermore, we provide conditions under which the optimal match-down-to levels become state-independent, which can be computationally desirable.

PROPOSITION 4. *The optimal match-down-to level $\bar{z}_j(t, x_1 - \tilde{y}_j, \mathbf{y}_{[j+1,m]})$ in the 1-to-m vertical model becomes a constant, independent of $x_1 - \tilde{y}_j$ and $\mathbf{y}_{[j+1,m]}$, if $\beta = 0$ or $\alpha = \beta$.*

We can obtain analogous results for the vertical model with n demand types and 1 supply type.

7. Bound and Heuristic: The Deterministic Model

In this section we study the deterministic counterpart of the stochastic problem in its general form. We show that the heuristic suggested by the deterministic model can be computed efficiently and is asymptotically optimal for the stochastic problem.

7.1. The Deterministic Heuristic

We consider the deterministic model by ignoring the uncertainty and assume that the mean demand quantity $\lambda_{it} = ED_{it}$ and mean supply quantity $\mu_{jt} = ES_{jt}$ arrive in each period. The linear program

$$
\begin{aligned}
\text{(P)} \quad \max_{q_{ijt}} \quad & \sum_{t=1}^{T} \gamma^{t-1}[\sum_{i=1}^{n}\sum_{j=1}^{m} r_{ij}q_{ijt} - c(\sum_{i=1}^{n} x_{it} - \sum_{i=1}^{n}\sum_{j=1}^{m} q_{ijt}) - h(\sum_{j=1}^{m} y_{jt} - \sum_{i=1}^{n}\sum_{j=1}^{m} q_{ijt})] \\
\text{s.t.} \quad & \sum_{j=1}^{m} q_{ijt} \le x_{it}, \quad 1 \le i \le n, 1 \le t \le T, \\
& \sum_{i=1}^{n} q_{ijt} \le y_{jt}, \quad 1 \le j \le m, 1 \le t \le T, \\
& x_{i,t+1} = \alpha(x_{it} - \sum_{j=1}^{m} q_{ijt}) + \lambda_{it}, \quad 1 \le i \le n, 1 \le t \le T-1, \\
& y_{j,t+1} = \beta(y_{jt} - \sum_{i=1}^{n} q_{ijt}) + \mu_{jt}, \quad 1 \le j \le m, 1 \le t \le T-1, \\
& q_{ijt} \ge 0, \quad 1 \le i \le n, 1 \le j \le m, 1 \le t \le T,
\end{aligned}
$$

gives the formulation of the problem, where $(\mathbf{x}_1, \mathbf{y}_1)$ is a given initial state.

We can rewrite the formulation (P) as a minimization problem and write its dual as a maximization problem as follows:

$$
\begin{aligned}
\text{(D)} \quad \max_{p_{it}^d, p_{jt}^s, f_{it}^d, f_{jt}^s} \quad & \sum_{t=1}^{T-1}\sum_{i=1}^{n} f_{it}^d \lambda_{it} + \sum_{t=1}^{T-1}\sum_{j=1}^{m} f_{jt}^s \mu_{jt} - \sum_{i=1}^{n} p_{i1}^d x_{i1} - \sum_{j=1}^{m} p_{j1}^s y_{j1} + \alpha \sum_{i=1}^{n} f_{i1}^d x_{i1} + \beta \sum_{j=1}^{m} f_{j1}^s y_{j1} \\
\text{s.t.} \quad & p_{it}^d + p_{jt}^s - \alpha f_{it}^d - \beta f_{jt}^s \ge \gamma^{t-1}(r_{ij} + c + h), \quad 1 \le i \le n, 1 \le j \le m, 1 \le t \le T, \\
& p_{it}^d - \alpha f_{it}^d + f_{i,t-1}^d = \gamma^{t-1} c, \quad 1 \le i \le n, 2 \le t \le T, \\
& p_{jt}^s - \beta f_{jt}^s + f_{j,t-1}^s = \gamma^{t-1} h, \quad 1 \le j \le m, 2 \le t \le T, \\
& p_{it}^d, p_{jt}^s \ge 0, \quad 1 \le i \le n, 1 \le j \le m, 1 \le t \le T,
\end{aligned}
$$

where $f_{iT}^d \equiv 0$ and $f_{jT}^s \equiv 0$ for $1 \le i \le n$ and $1 \le j \le m$.

The complementary slackness (CS) conditions for the above primal-dual pair are given as follows:

$$
\begin{aligned}
\text{(CS)} \quad & p_{it}^d(x_{it} - \sum_{j=1}^{m} q_{ijt}) = 0, \quad 1 \le i \le n, 1 \le t \le T, \\
& p_{jt}^s(y_{jt} - \sum_{i=1}^{n} q_{ijt}) = 0, \quad 1 \le j \le m, 1 \le t \le T, \\
& q_{ijt}[p_{it}^d + p_{jt}^s - \alpha f_{it}^d - \beta f_{jt}^s - \gamma^{t-1}(r_{ij} + c + h)] = 0, \quad 1 \le i \le n, 1 \le j \le m, 1 \le t \le T.
\end{aligned}
$$

By the strong duality, we immediately have the following result, which is standard, see, e.g., Sethi and Thompson (2005, section 8.2.2).

Proposition 5 (Solution to the deterministic model). *A matching decision* $\{\mathbf{Q}_t\}_{t=1,\ldots,T}$ *is optimal if and only if there exist dual variables* $\{p_{it}^d, p_{jt}^s\}_{1\le i\le n, 1\le t\le T}$, $\{f_{it}^d, f_{jt}^s\}_{1\le j\le m, 1\le t\le T-1}$, *with* $\{\mathbf{Q}_t\}_{t=1,\ldots,T}$, *such that the constraints of problems (P) and (D) and conditions (CS) are satisfied.*

The multipliers f_{it}^d, f_{jt}^s, $1\le i\le n$, $1\le j\le m$, $1\le t\le T$, correspond to the state transition equations in problem (P) on the demand and supply side respectively, which can be interpreted as the *shadow prices* of changing the system states. If these shadow prices are known or can be approximated by good proxies, e.g., the typical market prices that encourage demand and supply to enter the market for a given time period, then the optimal matching decisions for that period can be obtained or approximated by the following linear program for that period.

$$\text{(s-p)}\quad \max_{q_{ijt}} \quad \sum_{i=1}^{n}\sum_{j=1}^{m}(\gamma^{t-1}r_{ij}+\alpha f_{it}^d+\beta f_{jt}^s)q_{ijt}-\gamma^{t-1}c\Big(\sum_{i=1}^{n}x_{it}-\sum_{i=1}^{n}\sum_{j=1}^{m}q_{ijt}\Big)-\gamma^{t-1}h\Big(\sum_{j=1}^{m}y_{jt}-\sum_{i=1}^{n}\sum_{j=1}^{m}q_{ijt}\Big)$$
$$\text{s.t.}\quad \sum_{j=1}^{m}q_{ijt}\le x_{it},\quad 1\le i\le n;\quad \sum_{i=1}^{n}q_{ijt}\le y_{jt},\quad 1\le j\le m;\quad q_{ijt}\ge 0,\quad 1\le i\le n, 1\le j\le m.$$

The following corollary verifies that a necessary condition for an optimal matching decision $\mathbf{Q}_t$ in period t is that it solves the single-period matching problem (s-p).

Corollary 6. *The optimal matching decision* $\{\mathbf{Q}_t\}_{1\le t\le T}$ *solves the subproblem (s-p) for* $1\le t\le T$.

Intuitively, one would expect that the uncertainty in demand and supply in the stochastic model would result in lower expected surplus. One can think of the variables x_{it}, y_{jt} and q_{ijt} as the expected available quantity of type i demand, that of type j supply and the expected matching quantity of arc (i,j) between i and j in period t. While the stochastic problem requires constraints in (1) to be satisfied for each sample path, the deterministic model only requires the expected variables x_{it}, y_{jt} and q_{ijt} to satisfy the constraints. Thus the deterministic model is a relaxation of the stochastic one and provides an upper bound on the stochastic model's optimal surplus.

Proposition 6 (Deterministic upper bound). *The deterministic model provides an upper bound on the optimal total surplus of the stochastic model.*

Next we show that the heuristic policy suggested by the deterministic problem is asymptotic optimal. Consider a series of stochastic systems indexed by $k=1,2,\ldots$, with 1 representing the original system. We scale the time in system k so that the arrival of demand and supply in system k is k times more intense compared with the original system (equivalently, the clock is k times faster than the original system) in any given period t. Thus, instead of having random demand D_{it} for type $i\in\mathcal{D}$ and random supply S_{jt} for type $j\in\mathcal{S}$ arriving in a period, system k will have an amount of

$\sum_{\ell=1}^{k} D_{it}(\ell)$ for type $i \in \mathcal{D}$ and $\sum_{\ell=1}^{k} S_{jt}(\ell)$ for type $j \in \mathcal{S}$, where the $D_{it}(\ell)$'s are i.i.d. with the same distribution as D_{it}, and the $S_{jt}(\ell)$'s are i.i.d. with the same distribution as S_{jt}. We denote $\mathbf{D}_t^k = \sum_{\ell=1}^{k} \mathbf{D}_t(\ell) = (\sum_{\ell=1}^{k} D_{1t}(\ell), \ldots, \sum_{\ell=1}^{k} D_{nt}(\ell))$ and $\mathbf{S}_t^k = \sum_{\ell=1}^{k} \mathbf{S}_t(\ell) = (\sum_{\ell=1}^{k} S_{1t}(\ell), \ldots, \sum_{\ell=1}^{k} S_{mt}(\ell))$ as the demand and supply in system k, respectively. Note that $E\mathbf{D}_t^k = kE\mathbf{D}_t = k\boldsymbol{\lambda}_t$ and $E\mathbf{S}_t^k = kE\mathbf{S}_t = k\boldsymbol{\mu}_t$. Let $V_t^k(\mathbf{x}, \mathbf{y})$ be the value function in system k, and $V_t^{\text{det}}(\mathbf{x}, \mathbf{y})$ be the value function for the deterministic model. The functions $H_t^k(\mathbf{Q}, \mathbf{x}, \mathbf{y})$ and $H_t^{\text{det}}(\mathbf{Q}, \mathbf{x}, \mathbf{y})$ are defined accordingly for system k and the deterministic system, respectively. We have the following results that relate the stochastic systems with the deterministic model.

PROPOSITION 7. $\lim_{k\to\infty} V_t^k(k\mathbf{x}, k\mathbf{y})/k = V_t^{\text{det}}(\mathbf{x}, \mathbf{y})$ *for* $t = 1, 2, \ldots, T+1$, *and the convergence is uniform with respect to* $(\mathbf{x}, \mathbf{y})$.

We define a feasible decision $\mathbf{Q}$ under state $(\mathbf{x}, \mathbf{y})$ in period t as asymptotically optimal for the stochastic system if, for any $\epsilon > 0$, there exists $K > 0$ such that $H_t^k(k\mathbf{Q}, k\mathbf{x}, k\mathbf{y})/k \geq H_t^k(\mathbf{Q}(k)^*, k\mathbf{x}, k\mathbf{y})/k - \epsilon = V_t^k(k\mathbf{x}, k\mathbf{y})/k - \epsilon$ for all $k \geq K$, where $\mathbf{Q}(k)^*$ is an optimal policy for system k under the state $(k\mathbf{x}, k\mathbf{y})$.

THEOREM 8 (Asymptotic optimality of the deterministic heuristic). *The optimal matching policy policy* $\hat{\mathbf{Q}}_t(\mathbf{x}, \mathbf{y})$ *for the deterministic model, solved from the linear program (P) with* $T - t + 1$ *periods and* $\mathbf{x}_1 = \mathbf{x}$ *and* $\mathbf{y}_1 = \mathbf{y}$, *is asymptotically optimal for the stochastic system.*

In Online Appendix E, we test the performance of the deterministic heuristic with randomly generated system parameters, and numerically verify that the heuristic indeed performs very well for the stochastic problem.

8. Extensions

In the end, we discuss quite a few extensions of the base model.

Time-dependent parameters. All of our results can be readily extended to allow the holding costs c and h, carry-over rates α and β, and discount factor γ to be time-varying. For the extension of time-dependent reward parameters, we need to generalize condition (D) in the modified Monge partial order $(i,j) \succeq (i,j')$ as: for any time $0 \leq t_1 \leq t_2 \leq T$, $r_{ij}(t_1) + r_{i'j'}(t_2) \geq r_{ij'}(t_1) + r_{i'j}(t_2)$.

Type-dependent parameters. It is likely that different types of demand or supply have heterogeneous holding-cost and carry-over rates. This may not be an issue for geographic locations as horizontally differentiated types, e.g., the backlog cost to Amazon or Uber's unsatisfied customers can be independent of the customers' locations. Theorem 2 can be preserved if we relax the type-homogeneous-parameter assumption to allow type-dependent waiting and holding costs, as long as $c_i \geq c_{i'}$ for all $i' \in \mathcal{D}$ and $h_j \geq h_{j'}$ for all $j' \in \mathcal{S}$. Theorem 1 can be extended as long as the holding and waiting cost rates are ordered in alignment with the priority structure determined by the rewards. For example, in the vertical model of section 6, if the holding and waiting costs are type-dependent, the results carry over under the assumptions $c_1 \geq \cdots \geq c_n$ and $h_1 \geq \cdots \geq h_m$.

Random abandonment. After the matching in each period, unmatched demand and supply may abandon the wait with uncertainty. Our results will go through for random carry-over rates as long as they are realized the same for all types on each side.

Forbidden arcs. For notation convenience, we assume that matching between *any* pair of type i demand and type j supply is allowed in the base model. In reality, however, it is possible that a certain demand-supply pair (i,j) is undesirable or incompatible. We can consider a set of forbidden arcs $\mathcal{F} \subset \mathcal{A}$. The complement set $\mathcal{F}^c \stackrel{\text{def}}{=} \mathcal{A} \setminus \mathcal{F}$ is the permissible set of arcs along which matching is allowed. As long as the rewards satisfy the modified Monge conditions over the set $\mathcal{F}^c$, all of our results continue to hold under the restriction of forbidden arcs.

Forced maxing out. In practice, regulations, contracts, or social concerns may prevent the intermediary from deliberately saving demand and supply for future. Our priority results remain valid under the requirement of maxing out matching for any period. In fact, we can completely characterize the optimal matching policy under the restriction of forced maxing out, for those scenarios in which all neighboring arcs are comparable under the partial order (e.g., the horizontal and vertical models). The optimal policy is simply to match demand with supply in a greedy fashion according to the priority hierarchy, until either all the demand or all the supply is exhausted.

Other forms of reward structure. In addition to the reward structures studied in sections 5 and 6, our results can also be applied to other forms of reward structures to partially characterize the optimal matching policy in those settings. For example, in a horizontal circle model, if we consider the shortest distance rather than the unidirectional distance, a parallel version of Theorem 3 can be established. Suppose that r_{ij} is a linearly nonincreasing function of the shortest distance. Then, $(i,j) \succeq (i',j')$, if (i) the shortest path from i to j on the circle (i.e., the shorter circular segment between i and j) is a subset of the shortest path from i' to j' and (ii) the shortest-distance travel from j to i is in the same direction as the shortest-distance travel from j' to i'. However, the two types $i \in \mathcal{D}$ and $j \in \mathcal{S}$ that are closest to each other on the circle may not constitute a perfect pair anymore. In the vertical model, if $r_i^d \geq r_j^s \geq r_{j'}^s$ or $r_i^d \leq r_j^s \leq r_{j'}^s$, then $(i,j) \succeq (i,j')$ when $r_{ij} = \min\left\{r_i^d, r_j^s\right\}$, and $(i,j') \succeq (i,j)$ when $r_{ij} = \max\left\{r_i^d, r_j^s\right\}$.

Other setups. If the types are in a continuum, the priority results still hold. They would also carry over for an infinite-horizon periodic-review model with either discounted or long-run average payoff criterion, with $\alpha < 1$ and $\beta < 1$ required to ensure stability of the system. We expect the modified Monge conditions that guarantee the priority property for our periodic-review system also do so for a continuous-review system with demand and supply arrivals following a point process, which we leave to the future research.

9. Conclusion

We generalize, the Monge sequence condition that is necessary and sufficient for a greedy algorithm to solve a deterministic and balanced transportation problem, to a stochastic and dynamic matching problem. The generalization involves extending the notion of "augmenting path" to a stochastic and dynamic setting through backward induction. The modified Monge conditions on the reward matrix that we discover are sufficient, and in a robust sense, necessary, to guarantee a priority structure (weaker than the greedy algorithm) in the optimal matching policy for the general problem with intertemporally random demand and supply. We further present two underlying reward structures that satisfy the modified Monge condition for all neighboring pairs. Under the two specialized reward structures, along the priority hierarchy, when it comes to the matching between a specific pair, the optimal policy has a match-down-to threshold structure. This structure connects to the base stock policy in inventory management and protection levels in quantity-based revenue management. Though those two structures include many classic and emerging problems as special cases, many practical settings generally fail to satisfy the modified Monge condition. For example, in the vertical model, if the reward function is a general supermodular function other than an additive one, there exist scenarios in which socially efficient matching is not a top-down matching (i.e., assortative mating). In the horizontal model, if the distance is the shortest distance, there exist scenarios in which matching-to-the-closest is not optimal. For the general setting, we propose a deterministic heuristic that is asymptotically optimal and numerically performs well. Though suboptimal, many ride-sharing platforms still implement the "matching-to-the-closest" heuristic. It is desirable to demonstrate a worst-case performance bound for a heuristic policy, e.g., a greedy heuristic.

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Online Appendix to
"Dynamic Matching in a Two-Sided Market"

Ming Hu and Yun Zhou

A. The Partial Order

Lemma A.1. *If $(i,j)\succeq(i,j')$ and $(i,j')\succeq(i,j'')$, then $(i,j)\succeq(i,j'')$.*

Proof of Lemma A.1. $(i,j)\succeq(i,j')$ implies that $r_{ij}+r_{i'j'}\geq r_{ij'}+r_{i'j}$ for all $i'\in\mathcal{D}$ and $r_{ij}\geq r_{ij'}$. $(i,j')\succeq(i,j'')$ implies that $r_{ij'}+r_{i'j''}\geq r_{ij''}+r_{i'j'}$ for all $i'\in\mathcal{D}$ and $r_{ij'}\geq r_{ij''}$. Adding up the two inequalities leads to $r_{ij}+r_{i'j''}\geq r_{i'j}+r_{ij''}$ for all $i'\in\mathcal{D}$. Moreover, $r_{ij}\geq r_{ij''}$. It follows from Definition 1 that $(i,j)\succeq(i,j'')$. $\square$

Consider two arcs $(i,j)\succeq(i',j')$ and $i\neq i'$, $j\neq j'$. By Definition 2, there exists a decreasing sequence $(i_1,j_1)\succeq\cdots\succeq(i_n,j_n)$ connecting (i,j) and (i',j'). By virtue of Lemma A.1, we can assume without loss of generality that there are no three consecutive arcs sharing the same node. In fact, if $i_k=i_{k+1}=i_{k+2}$, then $(i_k,j_k)\succeq(i_{k+2},j_{k+2})$ by Lemma A.1. Thus we can remove (i_{k+1},j_{k+1}) from the sequence and the remaining arcs still constitute a decreasing sequence connecting (i,j) and (i',j'). Under the assumption that no three consecutive arcs share the same node, if (i_k,j_k) and (i_{k+1},j_{k+1}) share the same demand node $i_k=i_{k+1}$, (i_{k+1},j_{k+1}) and (i_{k+2},j_{k+2}) must share the same supply node $j_{k+1}=j_{k+2}$. So the sequence connecting (i,j) and (i',j') must follow a zigzag *path*.

Next, we show that for $(i,j)\succeq(i',j')$, $i\neq i'$ and $j\neq j'$, there exists a decreasing sequence connecting (i,j) and (i',j') that does not contain a cycle.

Lemma A.2. *Consider a decreasing sequence of arcs in the form $(i_1,j_1)\succeq(i_2,j_1)\succeq(i_2,j_2)\succeq\cdots\succeq(i_\ell,j_\ell)\succeq(i_1,j_\ell)$, with $i_1,\dots,i_\ell$ and $j_1,\dots,j_\ell$ being distinct nodes. Then, $(i_1,j_1)\succeq(i_1,j_\ell)$.*

Proof of Lemma A.2. Since for $k=1,\dots,l-1$, $(i_k,j_k)\succeq(i_{k+1},j_k)$, we have $r_{i_kj_k}+r_{i_{k+1}j'}\geq r_{i_kj'}+r_{i_{k+1}j_k}$ for all j'. Likewise, for $k=1,\dots,l-1$, $(i_{k+1},j_k)\succeq(i_{k+1},j_{k+1})$ implies that $r_{i_{k+1}j_k}+r_{i'j_{k+1}}\geq r_{i_{k+1}j_{k+1}}+r_{i'j_k}$ for all i', and $(i_\ell,j_\ell)\succeq(i_1,j_\ell)$ implies that $r_{i_\ell j_\ell}+r_{i_1j'}\geq r_{i_\ell j'}+r_{i_1j_\ell}$ for all j'. Sum up all these inequalities for any given i' and j',

$$\sum_{k=1}^{\ell-1}(r_{i_kj_k}+r_{i_{k+1}j'})+\sum_{k=1}^{\ell-1}(r_{i_{k+1}j_k}+r_{i'j_{k+1}})+r_{i_\ell j_\ell}+r_{i_1j'}\geq\sum_{k=1}^{\ell-1}(r_{i_kj'}+r_{i_{k+1}j_k})+\sum_{k=1}^{\ell-1}(r_{i_{k+1}j_{k+1}}+r_{i'j_k})+r_{i_\ell j'}+r_{i_1j_\ell}.$$

By rearranging terms, the left-hand-side of the above inequality can be rewritten as $\sum_{k=1}^{\ell}r_{i_kj_k}+\sum_{k=1}^{\ell}r_{i_kj'}+\sum_{k=1}^{\ell-1}r_{i_{k+1}j_k}+\sum_{k=2}^{\ell}r_{i'j_k}$, and the right-hand-side can be rewritten as $\sum_{k=1}^{\ell}r_{i_kj'}+\sum_{k=1}^{\ell-1}r_{i_{k+1}j_k}+\sum_{k=2}^{\ell}r_{i_kj_k}+\sum_{k=1}^{\ell-1}r_{i'j_k}+r_{i_1j_\ell}$. Then, it is easy to see that the inequality becomes $r_{i_1j_1}+r_{i'j_\ell}\geq r_{i'j_1}+r_{i_1j_\ell}$, for any i', after canceling out the common terms on both sides. Moreover, $(i_1,j_1)\succeq(i_2,j_1)\succeq(i_2,j_2)\succeq\cdots\succeq(i_\ell,j_\ell)\succeq(i_1,j_\ell)$ implies that $r_{i_1j_1}\geq r_{i_2j_1}\geq r_{i_2j_2}\geq\cdots\geq r_{i_\ell j_\ell}\geq r_{i_1j_\ell}$ and thus $r_{i_1j_1}\geq r_{i_1j_\ell}$. Therefore, $(i_1,j_1)\succeq(i_1,j_\ell)$. $\square$

By Lemma A.2, if $(i,j) \succeq (i',j')$, there exists a zigzag path of decreasing arcs that connects (i,j) and (i',j') and does not visit any node more than once. Finally, we verify the partial order.

B. Proofs

Proof of Proposition 1. We prove this result by induction on t. Clearly, $V_{T+1}(\mathbf{x},\mathbf{y}) \equiv 0$ is continous and concave in $(\mathbf{x},\mathbf{y})$. We suppose $V_{t+1}(\mathbf{x},\mathbf{y})$ is continous and concave in $(\mathbf{x},\mathbf{y})$, and show that so is $V_t(\mathbf{x},\mathbf{y})$. First, because $V_{t+1}(\mathbf{x},\mathbf{y})$ is continous in $(\mathbf{x},\mathbf{y})$, $H_t(\mathbf{Q},\mathbf{x},\mathbf{y})$ is continous in $(\mathbf{Q},\mathbf{x},\mathbf{y})$. Moreover, because the set mapping from $(\mathbf{x},\mathbf{y})$ to the set $\mathcal{R}(\mathbf{Q};\mathbf{x},\mathbf{y}) = \{\mathbf{Q} \mid \mathbf{Q} \geq 0, \mathbf{u} = \mathbf{x} - \mathbf{1}^m\mathbf{Q}^\intercal \geq 0, \mathbf{v} = \mathbf{y} - \mathbf{1}^n\mathbf{Q} \geq 0\}$ is compact-valued and continuous, by the maximum theorem, $V_t(\mathbf{x},\mathbf{y})$ is continous in $(\mathbf{x},\mathbf{y})$. Second, since the composition of a concave function and an affine function is still concave (Simchi-Levi et al. 2014, Proposition 2.1.3(b)), $V_{t+1}(\alpha\mathbf{u}+\mathbf{D},\beta\mathbf{v}+\mathbf{S})$ is concave in $(\mathbf{Q},\mathbf{x},\mathbf{y})$ for any given $(\mathbf{D},\mathbf{S})$. Then, $E_{(\mathbf{D},\mathbf{S})}[V_{t+1}(\alpha\mathbf{u}+\mathbf{D},\beta\mathbf{v}+\mathbf{S})]$ is concave in $(\mathbf{Q},\mathbf{x},\mathbf{y})$. Then it is immediately clear that $H_t(\mathbf{Q},\mathbf{x},\mathbf{y})$ is jointly concave in $(\mathbf{Q},\mathbf{x},\mathbf{y})$, because all other terms except the last term in (1) are linear in $(\mathbf{Q},\mathbf{x},\mathbf{y})$. Because the set $\mathcal{R}(\mathbf{Q};\mathbf{x},\mathbf{y})$ is a polyhedron defined by a system of linear inequalities, and a fortiori, a convex set, and the concavity is preserved under maximization over a convex set (Simchi-Levi et al. 2014, Proposition 2.1.15(b)), we have $V_t(\mathbf{x},\mathbf{y}) = \max_{\mathbf{Q}\in\mathcal{R}(\mathbf{Q};\mathbf{x},\mathbf{y})} H_t(\mathbf{Q},\mathbf{x},\mathbf{y})$ is concave.

The existence of an optimal matching policy $\mathbf{Q}_t^*(\mathbf{x},\mathbf{y})$ follows from the continuity of the function $H_t(\mathbf{Q},\mathbf{x},\mathbf{y})$ and the compactness of $\mathcal{R}(\mathbf{Q};\mathbf{x},\mathbf{y})$ for a given $(\mathbf{x},\mathbf{y})$. $\square$

Proof of Lemma 1. (Reflexivity) This follows directly from Definition 1.

(Antisymmetry) This follows from the definition of the equivalence relation $\simeq$.

(Transitivity) $\rho_1 \succeq \rho_2$ implies that there exists a decreasing sequence of arcs connecting ρ_1 and ρ_2, containing no cycles. Likewise, $\rho_2 \succeq \rho_3$ implies that there exists a decreasing sequence of arcs connecting ρ_2 and ρ_3, containing no cycles. Combining these two sequences, we have a decreasing sequence of arcs connecting ρ_1 and ρ_3.

If ρ_1 and ρ_3 share no nodes, then by Definition 2, $\rho_1 \succeq \rho_3$.

Now consider the case in which ρ_1 and ρ_3 have a common node. Let us assume without loss of generality that $\rho_1 = (i_1, j_1)$ and $\rho_3 = (i_1, j_3)$. If ρ_2 also shares the same node with them, i.e., node i_1, then by Lemma A.1, $\rho_1 \succeq \rho_3$.

It remains to check the case in which ρ_2 is not adjacent to the common node i_1 shared by ρ_1 and ρ_3. Note that there exists a decreasing zigzag path of arcs connecting ρ_2 and ρ_3 that does not visit any node on the path more than once. Consider the arc ρ immediately before ρ_3. The arc ρ is adjacent to either node j_3 or node i_1. In the former case, by Definition 2, the relation $\rho_1 \succeq \rho$ holds, because ρ_1 and ρ share no common nodes and there is a decreasing sequence of arcs connecting ρ_1 and ρ (consider the decreasing sequence connecting ρ_1 and ρ_3 with ρ_3 removed). Together with the arc ρ_3, the zigzag path connecting ρ_1 and ρ forms a cycle. By Lemma A.2, $\rho_1 \succeq \rho_3$.

If the latter case, i.e., ρ is adjacent to i_1, then $\rho = (i_1, j)$ for some j and there is an arc $\rho' = (i, j)$ immediately before ρ in the decreasing path that connects ρ_2 and ρ_3, with $i \neq i_1$. Since there clearly exists a decreasing sequence connecting ρ_1 and ρ', which do not share a common node, we have $\rho_1 \succeq \rho'$. Then, together with the arc ρ, the zigzag path connecting ρ_1 and ρ' forms a cycle. Again by Lemma A.2, $\rho_1 \succeq \rho$. Since $\rho \succeq \rho_3$ and all of ρ_1, ρ and ρ_3 share the same node i_1, it follows from Lemma A.1 that $\rho_1 \succeq \rho_3$. $\square$

Proof of Lemma 3. The result holds trivially for $t = T+1$ by the boundary condition $V_{T+1}(\mathbf{x}, \mathbf{y}) \equiv 0$. Suppose it holds for $t+1$. We show that it also holds for t.

Consider a given $(\mathbf{x}, \mathbf{y})$ with $x_i > 0$ and $y_j > 0$, $\epsilon_t^1 \in [0, x_i]$ and $\epsilon_t^2 \in [0, y_j]$. Let $\hat{\mathbf{Q}} \in \arg\max_{\mathbf{Q} \in \{\mathbf{Q} \geq \mathbf{0} | \mathbf{u} \geq \mathbf{0}, \mathbf{v} \geq \mathbf{0}\}} H_t(\mathbf{Q}, \mathbf{x}, \mathbf{y})$. We claim that:

Claim. There exist nonnegative numbers $\eta_{j''}^t$ for $j'' \in \mathcal{S}$ and $\xi_{i''}^t$ for $i'' \in \mathcal{D}$ such that $\sum_{j'' \in \mathcal{S}} \eta_{j''}^t \leq \epsilon_t^1$, $\sum_{i'' \in \mathcal{D}} \xi_{i''}^t \leq \epsilon_t^2$, and the decision $\widetilde{\mathbf{Q}} = \hat{\mathbf{Q}} + \sum_{i'' \in \mathcal{D}} (\xi_{i''}^t \mathbf{e}_{i''j'}^{n \times m} - \xi_{i''}^t \mathbf{e}_{i''j}^{n \times m}) + \sum_{j'' \in \mathcal{S}} (\eta_{j''}^t \mathbf{e}_{i'j''}^{n \times m} - \eta_{j''}^t \mathbf{e}_{ij''}^{n \times m})$ is feasible under the state $(\mathbf{x} - \epsilon_t^1 \mathbf{e}_i^n + \epsilon_t^1 \mathbf{e}_{i'}^n, \mathbf{y} - \epsilon_t^2 \mathbf{e}_j^m + \epsilon_t^2 \mathbf{e}_{j'}^m)$.

Proof of Claim. We construct $\eta_{j''}^t$ as follows. Let $\eta_1^t = \min\{\hat{q}_{i1}, \epsilon_t^1\}$. Then, recursively, let $\eta_{j''}^t = \min\left\{\hat{q}_{ij''}, \epsilon_t^1 - \sum_{k=1}^{j''-1} \eta_k^t\right\}$ for $j'' = 2, \ldots, m$.

We first prove $\epsilon_t^1 - \sum_{k=1}^{j''} \eta_k^t = (\epsilon_t^1 - \sum_{k=1}^{j''} \hat{q}_{ik})^+$ for all $j'' \in \mathcal{S}$ by induction, which guarantees that $\eta_{j''}^t \geq 0$ and $\sum_{j''=1}^m \eta_{j''}^t \leq \epsilon_t^1$. For $j = 1$, $\epsilon_t^1 - \eta_1^t = \epsilon_t^1 - \min\{\hat{q}_{i1}, \epsilon_t^1\} = (\epsilon_t^1 - \hat{q}_{i1})^+$. Thus the equation holds for $j'' = 1$. Suppose it holds for j''. Then for $j'' + 1$,

$$
\begin{aligned}
\epsilon_t^1 - \sum_{k=1}^{j''+1} \eta_k^t =& (\epsilon_t^1 - \sum_{k=1}^{j''} \eta_k^t) - \eta_{j''+1}^t = (\epsilon_t^1 - \sum_{k=1}^{j''} \eta_k^t) - \min\{\epsilon_t^1 - \sum_{k=1}^{j''} \eta_k^t, \hat{q}_{i,j''+1}\} \\
=& (\epsilon_t^1 - \sum_{k=1}^{j''} \hat{q}_{ik})^+ - \min\{(\epsilon_t^1 - \sum_{k=1}^{j''} \hat{q}_{ik})^+, \hat{q}_{i,j''+1}\} \\
=& [(\epsilon_t^1 - \sum_{k=1}^{j''} \hat{q}_{ik})^+ - \hat{q}_{i,j''+1}]^+ = [\epsilon_t^1 - \sum_{k=1}^{j''} \hat{q}_{ik} - \hat{q}_{i,j''+1}]^+ = [\epsilon_t^1 - \sum_{k=1}^{j''+1} \hat{q}_{ik}]^+,
\end{aligned}
$$

which completes the induction.

Hence, we have $\epsilon_t^1 - \sum_{j''=1}^m \eta_{j''}^t = (\epsilon_t^1 - \sum_{j''=1}^m \hat{q}_{ij''})^+$. Case (i): If $\sum_{j''=1}^m \hat{q}_{ij''} < \epsilon_t^1$, then $\epsilon_t^1 - \sum_{j''=1}^m \eta_{j''}^t = \epsilon_t^1 - \sum_{j''=1}^m \hat{q}_{ij''}$, implying that $\sum_{j''=1}^m \hat{q}_{ij''} = \sum_{j''=1}^m \eta_{j''}^t$. Case (ii): If $\sum_{j''=1}^m \hat{q}_{ij''} \geq \epsilon_t^1$, then $\epsilon_t^1 - \sum_{j''=1}^m \eta_{j''}^t = 0$, implying that $\epsilon_t^1 = \sum_{j''=1}^m \eta_{j''}^t$. Combining the two cases, we have $\sum_{j''=1}^m \eta_{j''}^t \leq \epsilon_t^1$.

Now we show that the decision $\bar{\mathbf{Q}} = \hat{\mathbf{Q}} + \sum_{j'' \in \mathcal{S}} (\eta_{j''}^t \mathbf{e}_{i'j''}^{n \times m} - \eta_{j''}^t \mathbf{e}_{ij''}^{n \times m})$ is feasible under the state $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \equiv (\mathbf{x} - \epsilon_t^1 \mathbf{e}_i^n + \epsilon_t^1 \mathbf{e}_{i'}^n, \mathbf{y})$ in period t.

Since $\hat{\mathbf{Q}}$ is optimal for the state $(\mathbf{x},\mathbf{y})$, a fortiori, $\hat{\mathbf{Q}}$ is feasible, i.e., $\hat{\mathbf{Q}} \geq 0$, $\mathbf{1}^m\hat{\mathbf{Q}}^\intercal \leq \mathbf{x}$ and $\mathbf{1}^n\hat{\mathbf{Q}} \leq \mathbf{y}$. We show that $\bar{\mathbf{Q}}$ is feasible for the new state $(\bar{\mathbf{x}},\bar{\mathbf{y}}) \geq \mathbf{0}$, where the latter inequality is due to $x_i > 0$ and $\epsilon_t^1 \in [0, x_i]$. To this end, it suffices to show that $\bar{\mathbf{Q}} \geq 0$, $\mathbf{1}^m\bar{\mathbf{Q}}^\intercal \leq \bar{\mathbf{x}}$ and $\mathbf{1}^n\bar{\mathbf{Q}} \leq \bar{\mathbf{y}}$.

First, for all j, because $0 \leq \eta_j^t \leq \hat{q}_{ij}$, we have $\bar{q}_{ij} = \hat{q}_{ij} - \eta_j^t \geq 0$. Also, it is clear that $\bar{q}_{i'j} = \hat{q}_{i'j} + \eta_j^t \geq 0$ for all j. For any $i'' \neq i, i'$, we have $\bar{q}_{i''j} = \hat{q}_{i''j} \geq 0$ for all j. Thus, $\bar{\mathbf{Q}} \geq 0$.

Second, we have $\mathbf{1}^m\bar{\mathbf{Q}}_i^\intercal = \mathbf{1}^m\hat{\mathbf{Q}}_i^\intercal - \sum_{j''\in\mathcal{S}} \eta_{j''}^t = \sum_{j'=1}^m \hat{q}_{ij'} - \sum_{j''\in\mathcal{S}} \eta_{j''}^t$. If $\sum_{j=1}^m \hat{q}_{ij} < \epsilon_t^1$, we have $\mathbf{1}^m\bar{\mathbf{Q}}_i^\intercal = \sum_{j'=1}^m \hat{q}_{ij'} - \sum_{j''\in\mathcal{S}} \eta_{j''}^t = 0 \leq \bar{x}_i$. If $\sum_{j=1}^m \hat{q}_{ij} \geq \epsilon_t^1$, then $\epsilon_t^1 = \sum_{j''\in\mathcal{S}} \eta_{j''}^t$. Thus, $\mathbf{1}^m\bar{\mathbf{Q}}_i^\intercal = \mathbf{1}^m\hat{\mathbf{Q}}_i^\intercal - \sum_{j''\in\mathcal{S}} \eta_{j''}^t = \mathbf{1}^m\hat{\mathbf{Q}}_i^\intercal - \epsilon_t^1 \leq x_i - \epsilon_t^1 = \bar{x}_i$. We also have $\mathbf{1}^m\bar{\mathbf{Q}}_{i'}^\intercal = \mathbf{1}^m\hat{\mathbf{Q}}_{i'}^\intercal + \sum_{j''\in\mathcal{S}} \eta_{j''}^t \leq \mathbf{1}^m\hat{\mathbf{Q}}_{i'}^\intercal + \epsilon_t^1 \leq x_{i'} + \epsilon_t^1 = \bar{x}_{i'}$. For any $i'' \neq i, i'$, $\mathbf{1}^m\bar{\mathbf{Q}}_{i''}^\intercal = \mathbf{1}^m\hat{\mathbf{Q}}_{i''}^\intercal \leq x_{i''} = \bar{x}_{i''}$. Therefore, $\mathbf{1}^m\bar{\mathbf{Q}}^\intercal \leq \bar{\mathbf{x}}$.

Finally, $\mathbf{1}^n\bar{\mathbf{Q}} = \mathbf{1}^n\hat{\mathbf{Q}} + \mathbf{1}^n(-\sum_{j'=1}^m \eta_{j'}^t \mathbf{e}_{ij'}^{n\times m} + \sum_{j'=1}^m \eta_{j'}^t \mathbf{e}_{i'j'}^{n\times m}) = \mathbf{1}^n\hat{\mathbf{Q}} + 0 \leq \mathbf{y} = \bar{\mathbf{y}}$.

Define $\xi_1^t = \min\{\bar{q}_{1j}, \epsilon_t^2\}$ and $\xi_{i''}^t = \min\left\{\bar{q}_{i''j}, \epsilon_t^2 - \sum_{k=1}^{i''-1} \xi_k^t\right\}$ for $i'' = 2, \ldots, n$. Following a symmetric analysis, we can show that the decision $\widetilde{\mathbf{Q}} = \bar{\mathbf{Q}} + \sum_{i''\in\mathcal{D}}(\xi_{i''}^t \mathbf{e}_{i''j'}^{n\times m} - \xi_{i''}^t \mathbf{e}_{i''j}^{n\times m})$ is feasible under the state $(\mathbf{x} - \epsilon_t^1\mathbf{e}_i^n + \epsilon_t^1\mathbf{e}_{i'}^n, \mathbf{y} - \epsilon_t^2\mathbf{e}_j^m + \epsilon_t^2\mathbf{e}_{j'}^m) = (\bar{\mathbf{x}}, \bar{\mathbf{y}} - \epsilon_t^2\mathbf{e}_j^m + \epsilon_t^2\mathbf{e}_{j'}^m)$. This proves the claim. $\square$

Now denote by $\mathbf{u}$ and $\mathbf{v}$ the post-matching levels under the state $(\mathbf{x},\mathbf{y})$ and the decision $\hat{\mathbf{Q}}$ in period t. Define $\epsilon_{t+1}^1 = \alpha(\epsilon_t^1 - \sum_{j''\in\mathcal{S}} \eta_{j''}^t)$ and $\epsilon_{t+1}^2 = \beta(\epsilon_t^2 - \sum_{i''\in\mathcal{D}} \xi_{i''}^t)$. We have:

$$\begin{aligned}
&V_t(\mathbf{x} - \epsilon_t^1\mathbf{e}_i^n + \epsilon_t^1\mathbf{e}_{i'}^n, \mathbf{y} - \epsilon_t^2\mathbf{e}_j^m + \epsilon_t^2\mathbf{e}_{j'}^m) - V_t(\mathbf{x},\mathbf{y}) \\
\geq &H_t(\widetilde{\mathbf{Q}}, \mathbf{x} - \epsilon_t^1\mathbf{e}_i^n + \epsilon_t^1\mathbf{e}_{i'}^n, \mathbf{y} - \epsilon_t^2\mathbf{e}_j^m + \epsilon_t^2\mathbf{e}_{j'}^m) - H_t(\hat{\mathbf{Q}}, \mathbf{x}, \mathbf{y}) \\
\geq &\sum_{i''\in\mathcal{D}} \xi_{i''}^t(r_{i''j'} - r_{i''j}) + \sum_{j''\in\mathcal{S}} \eta_{j''}^t(r_{i'j''} - r_{ij''}) \\
&+ \gamma EV_{t+1}(\alpha[\mathbf{u} - (\epsilon_t^1 - \sum_{j''\in\mathcal{S}} \eta_{j''}^t)\mathbf{e}_i^n + (\epsilon_t^1 - \sum_{j''\in\mathcal{S}} \eta_{j''}^t)\mathbf{e}_{i'}^n] + \mathbf{D}, \beta[\mathbf{v} - (\epsilon_t^2 - \sum_{i''\in\mathcal{D}} \xi_{i''}^t)\mathbf{e}_j^m + (\epsilon_t^2 - \sum_{i''\in\mathcal{D}} \xi_{i''}^t)\mathbf{e}_{j'}^m] + \mathbf{S}) \\
&- \gamma EV_{t+1}(\alpha\mathbf{u} + \mathbf{D}, \beta\mathbf{v} + \mathbf{S}) \\
= &\sum_{i''\in\mathcal{D}} \xi_{i''}^t(r_{i''j'} - r_{i''j}) + \sum_{j''\in\mathcal{S}} \eta_{j''}^t(r_{i'j''} - r_{ij''}) \\
&+ \gamma EV_{t+1}(\alpha\mathbf{u} - \epsilon_{t+1}^1\mathbf{e}_i^n + \epsilon_{t+1}^1\mathbf{e}_{i'}^n + \mathbf{D}, \beta\mathbf{v} - \epsilon_{t+1}^2\mathbf{e}_j^m + \epsilon_{t+1}^2\mathbf{e}_{j'}^m + \mathbf{S}) - \gamma EV_{t+1}(\alpha\mathbf{u} + \mathbf{D}, \beta\mathbf{v} + \mathbf{S}). \quad (1)
\end{aligned}$$

Let $\mathbf{X}_{t+1} = \alpha\mathbf{u} + \mathbf{D}$ and $\mathbf{Y}_{t+1} = \beta\mathbf{v} + \mathbf{S}$. By the induction hypothesis, there exist $K_{j''}^\tau$ and $L_{i''}^\tau$ for $j'' \in \mathcal{S}$, $i'' \in \mathcal{D}$ and $\tau = t+1, \ldots, T$ such that $\sum_{\tau=t+1}^T \sum_{j''\in\mathcal{S}} K_{j''}^\tau \leq \epsilon_{t+1}^1$, $\sum_{\tau=t+1}^T \sum_{i''\in\mathcal{D}} L_{i''}^\tau \leq \epsilon_{t+1}^2$ and

$$\begin{aligned}
&V_{t+1}(\mathbf{X}_{t+1} - \epsilon_{t+1}^1\mathbf{e}_i^n + \epsilon_{t+1}^1\mathbf{e}_{i'}^n, \mathbf{Y}_{t+1} - \epsilon_{t+1}^2\mathbf{e}_j^m + \epsilon_{t+1}^2\mathbf{e}_{j'}^m) - V_{t+1}(\mathbf{X}_{t+1}, \mathbf{Y}_{t+1}) \\
&\geq \sum_{\tau=t+1}^T \gamma^{\tau-t-1}[\sum_{j''\in\mathcal{S}} K_{j''}^\tau(r_{i'j''} - r_{ij''}) + \sum_{i''\in\mathcal{D}} L_{i''}^\tau(r_{i''j'} - r_{i''j})]. \qquad (2)
\end{aligned}$$

Let $\eta_{j''}^\tau = EK_{j''}^\tau$ and $\xi_{i''}^\tau = EL_{i''}^\tau$ for $j'' \in \mathcal{S}$, $i'' \in \mathcal{D}$ and $\tau = t+1, \ldots, T$. We have

$$V_t(\mathbf{x} - \epsilon_t^1\mathbf{e}_i^n + \epsilon_t^1\mathbf{e}_{i'}^n, \mathbf{y} - \epsilon_t^2\mathbf{e}_j^m + \epsilon_t^2\mathbf{e}_{j'}^m) - V_t(\mathbf{x},\mathbf{y})$$

$$
\begin{aligned}
&\geq \sum_{i''\in\mathcal{D}} \xi^t_{i''}(r_{i''j'}-r_{i''j}) + \sum_{j''\in\mathcal{S}} \eta^t_{j''}(r_{i'j''}-r_{ij''})\\
&\quad + \gamma E V_{t+1}(\alpha\mathbf{u}-\epsilon^1_{t+1}\mathbf{e}^n_i+\epsilon^1_{t+1}\mathbf{e}^n_{i'}+\mathbf{D},\beta\mathbf{v}-\epsilon^2_{t+1}\mathbf{e}^m_j+\epsilon^2_{t+1}\mathbf{e}^m_{j'}+\mathbf{S})-\gamma E V_{t+1}(\alpha\mathbf{u}+\mathbf{D},\beta\mathbf{v}+\mathbf{S})\\
&\geq \sum_{i''\in\mathcal{D}} \xi^t_{i''}(r_{i''j'}-r_{i''j}) + \sum_{j''\in\mathcal{S}} \eta^t_{j''}(r_{i'j''}-r_{ij''}) + \sum_{\tau=t+1}^{T}\gamma^{\tau-t}[\sum_{j''\in\mathcal{S}}\eta^\tau_{j''}(r_{i'j''}-r_{ij''})+\sum_{i''\in\mathcal{D}}\xi^\tau_{i''}(r_{i''j'}-r_{i''j})]\\
&= \sum_{\tau=t}^{T}\gamma^{\tau-t}[\sum_{j''\in\mathcal{S}}\eta^\tau_{j''}(r_{i'j''}-r_{ij''})+\sum_{i''\in\mathcal{D}}\xi^\tau_{i''}(r_{i''j'}-r_{i''j})],
\end{aligned}
$$

where the first inequality is (1) and the second inequality is due to (2).

Moreover, $\sum_{\tau=t}^{T}\sum_{j''\in\mathcal{S}}\eta^\tau_{j''}=\sum_{j''\in\mathcal{S}}\eta^t_{j''}+E\sum_{\tau=t+1}^{T}\sum_{j''\in\mathcal{S}}K^\tau_{j''}\leq\sum_{j''\in\mathcal{S}}\eta^t_{j''}+\epsilon^1_{t+1}=\sum_{j''\in\mathcal{S}}\eta^t_{j''}+\alpha(\epsilon^1_t-\sum_{j''\in\mathcal{S}}\eta^t_{j''})=\alpha\epsilon^1_t+(1-\alpha)\sum_{j''\in\mathcal{S}}\eta^t_{j''}\leq\epsilon^1_t$, because $\sum_{j''\in\mathcal{S}}\eta^t_{j''}\leq\epsilon^1_t$. Similarly, $\sum_{\tau=t}^{T}\sum_{i''\in\mathcal{D}}\xi^\tau_{i''}=\sum_{i''\in\mathcal{D}}\xi^t_{i''}+E\sum_{\tau=t+1}^{T}\sum_{i''\in\mathcal{D}}L^\tau_{i''}=\sum_{i''\in\mathcal{D}}\xi^t_{i''}+\epsilon^2_{t+1}\leq\sum_{i''\in\mathcal{D}}\xi^t_{i''}+\beta(\epsilon^2_t-\sum_{i''\in\mathcal{D}}\xi^t_{i''})\leq\epsilon^2_t$. This completes the induction. □

Proof of Lemma 4. By Lemma 3, there exist nonnegative numbers $\eta^\tau_{j''}$ and $\xi^\tau_{i''}$ for $j''\in\mathcal{S}$, $i''\in\mathcal{D}$ and $\tau=t,\ldots,T+1$ such that $\sum_{\tau=t}^{T}\sum_{j''\in\mathcal{S}}\eta^\tau_{j''}\leq\epsilon_1$, $\sum_{\tau=t}^{T}\sum_{i''\in\mathcal{D}}\xi^\tau_{i''}\leq\epsilon_2$ and

$$
V_t(\mathbf{x}-\epsilon_1\mathbf{e}^n_i+\epsilon_1\mathbf{e}^n_{i'},\mathbf{y}-\epsilon_2\mathbf{e}^m_j+\epsilon_2\mathbf{e}^m_{j'})-V_t(\mathbf{x},\mathbf{y})\geq\sum_{\tau=t}^{T}\gamma^{\tau-t}[\sum_{j''\in\mathcal{S}}\eta^\tau_{j''}(r_{i'j''}-r_{ij''})+\sum_{i''\in\mathcal{D}}\xi^\tau_{i''}(r_{i''j'}-r_{i''j})]. \tag{3}
$$

Since $(i,j)\succeq(i',j')$, there exists a decreasing sequence connecting the two arcs. Without loss of generality, we choose a path in the form of $(i,j)=(i_1,j_1)\succeq(i_1,j_2)\succeq(i_2,j_2)\succeq\cdots\succeq(i_\ell,j_\ell)=(i',j')$, and the proof for the other forms would be analogous. The condition $(i_k,j_k)\succeq(i_{k+1},j_k)$ implies that $r_{i_kj_k}+r_{i_{k+1}j''}\geq r_{i_kj''}+r_{i_{k+1}j_k}$, i.e., $r_{i_{k+1}j''}-r_{i_kj''}\geq r_{i_{k+1}j_k}-r_{i_kj_k}$. Thus

$$
r_{i'j''}-r_{ij''}=r_{i_\ell j''}-r_{i_1j''}=\sum_{k=1}^{\ell-1}(r_{i_{k+1}j''}-r_{i_kj''})\geq\sum_{k=1}^{\ell-1}(r_{i_{k+1}j_k}-r_{i_kj_k}). \tag{4}
$$

Likewise, the condition $(i_{k+1},j_k)\succeq(i_{k+1},j_{k+1})$ implies that

$$
r_{i''j'}-r_{i''j}=r_{i''j_\ell}-r_{i''j_1}=\sum_{k=1}^{\ell-1}(r_{i''j_{k+1}}-r_{i''j_k})\geq\sum_{k=1}^{\ell-1}(r_{i_{k+1}j_{k+1}}-r_{i_{k+1}j_k}). \tag{5}
$$

Then,

$$
\begin{aligned}
&V_t(\mathbf{x}-\epsilon\mathbf{e}^n_i+\epsilon\mathbf{e}^n_{i'},\mathbf{y}-\epsilon\mathbf{e}^m_j+\epsilon\mathbf{e}^m_{j'})-V_t(\mathbf{x},\mathbf{y})\\
\geq&\sum_{\tau=t}^{T}\gamma^{\tau-t}\sum_{j''\in\mathcal{S}}\eta^\tau_{j''}\sum_{k=1}^{\ell-1}(r_{i_{k+1}j_k}-r_{i_kj_k})+\sum_{\tau=t}^{T}\gamma^{\tau-t}\sum_{i''\in\mathcal{D}}\xi^\tau_{i''}\sum_{k=1}^{\ell-1}(r_{i_{k+1}j_{k+1}}-r_{i_{k+1}j_k})\\
=&(\sum_{\tau=t}^{T}\gamma^{\tau-t}\sum_{j''\in\mathcal{S}}\eta^\tau_{j''}-\sum_{\tau=t}^{T}\gamma^{\tau-t}\sum_{i''\in\mathcal{D}}\xi^\tau_{i''})\sum_{k=1}^{\ell-1}(r_{i_{k+1}j_k}-r_{i_kj_k})\\
&+\sum_{\tau=t}^{T}\gamma^{\tau-t}\sum_{i''\in\mathcal{D}}\xi^\tau_{i''}[\sum_{k=1}^{\ell-1}(r_{i_{k+1}j_k}-r_{i_kj_k})+\sum_{k=1}^{\ell-1}(r_{i_{k+1}j_{k+1}}-r_{i_{k+1}j_k})]
\end{aligned}
$$

$$=(\sum_{\tau=t}^{T}\gamma^{\tau-t}\sum_{j''\in\mathcal{S}}\eta^{\tau}_{j''}-\sum_{\tau=t}^{T}\gamma^{\tau-t}\sum_{i''\in\mathcal{D}}\xi^{\tau}_{i''})\sum_{k=1}^{\ell-1}(r_{i_{k+1}j_k}-r_{i_kj_k})+\sum_{\tau=t}^{T}\gamma^{\tau-t}\sum_{i''\in\mathcal{D}}\xi^{\tau}_{i''}(r_{i_\ell j_\ell}-r_{i_1j_1}), \quad (6)$$

where the first inequality is due to (3), (4) and (5). Moreover,

$$\sum_{k=1}^{\ell-1}(r_{i_{k+1}j_k}-r_{i_kj_k})=r_{i_\ell j_\ell}-r_{i_1j_1}+\sum_{k=1}^{\ell-1}(r_{i_{k+1}j_k}-r_{i_{k+1}j_{k+1}})\geq r_{i_\ell j_\ell}-r_{i_1j_1}, \quad (7)$$

where the inequality follows from the condition $(i_{k+1},j_k)\succeq(i_{k+1},j_{k+1})$ that implies $r_{i_{k+1}j_k}-r_{i_{k+1}j_{k+1}}\geq 0$. Without loss of generality, let us assume $\sum_{\tau=t}^{T}\gamma^{\tau-t}\sum_{j''\in\mathcal{S}}\eta^{\tau}_{j''}\geq\sum_{\tau=t}^{T}\gamma^{\tau-t}\sum_{i''\in\mathcal{D}}\xi^{\tau}_{i''}$. Consequently,

$$V_t(\mathbf{x}-\epsilon_1\mathbf{e}^n_i+\epsilon_1\mathbf{e}^n_{i'},\mathbf{y}-\epsilon_2\mathbf{e}^m_j+\epsilon_2\mathbf{e}^m_{j'})-V_t(\mathbf{x},\mathbf{y})$$
$$\geq\sum_{\tau=t}^{T}\gamma^{\tau-t}\sum_{j''\in\mathcal{S}}\eta^{\tau}_{j''}(r_{i_\ell j_\ell}-r_{i_1j_1})\geq\sum_{\tau=t}^{T}\sum_{j''\in\mathcal{S}}\eta^{\tau}_{j''}(r_{i_\ell j_\ell}-r_{i_1j_1})\geq\epsilon_1(r_{i_\ell j_\ell}-r_{i_1j_1})\geq\epsilon(r_{i_\ell j_\ell}-r_{i_1j_1})=\epsilon(r_{i'j'}-r_{ij}),$$

where the first inequality is due to (6), (7) and the assumption that $\sum_{\tau=t}^{T}\gamma^{\tau-t}\sum_{j''\in\mathcal{S}}\eta^{\tau}_{j''}\geq\sum_{\tau=t}^{T}\gamma^{\tau-t}\sum_{i''\in\mathcal{D}}\xi^{\tau}_{i''}$, and the remaining inequalities are due to $r_{i_\ell j_\ell}=r_{i'j'}\leq r_{ij}=r_{i_1j_1}$ implied by $(i,j)\succeq(i',j')$. The first inequality can be shown similarly for the case $\sum_{\tau=t}^{T}\gamma^{\tau-t}\sum_{j''\in\mathcal{S}}\eta^{\tau}_{j''}<\sum_{\tau=t}^{T}\gamma^{\tau-t}\sum_{i''\in\mathcal{D}}\xi^{\tau}_{i''}$, with (6) rewritten in terms of $(\sum_{\tau=t}^{T}\gamma^{\tau-t}\sum_{i''\in\mathcal{D}}\xi^{\tau}_{i''}-\sum_{\tau=t}^{T}\gamma^{\tau-t}\sum_{j''\in\mathcal{S}}\eta^{\tau}_{j''})$. □

Proof of Lemma 2. Let $\mathbf{u}=\mathbf{x}-\mathbf{1}^m\mathbf{Q}^\intercal$ and $\mathbf{v}=\mathbf{y}-\mathbf{1}^n\mathbf{Q}$. We have

$$H_t(\mathbf{Q}+\epsilon\mathbf{e}^{n\times m}_{ij}-\epsilon\mathbf{e}^{n\times m}_{i'j'},\mathbf{x},\mathbf{y})-H_t(\mathbf{Q},\mathbf{x},\mathbf{y})$$
$$=(r_{ij}-r_{i'j'})\epsilon+\gamma EV_{t+1}(\alpha\mathbf{u}+\mathbf{D}-\alpha\epsilon\mathbf{e}^n_i+\alpha\epsilon\mathbf{e}^n_{i'},\beta\mathbf{v}+\mathbf{S}-\beta\epsilon\mathbf{e}^m_j+\beta\epsilon\mathbf{e}^m_{j'})-\gamma EV_{t+1}(\alpha\mathbf{u}+\mathbf{D},\beta\mathbf{v}+\mathbf{S})$$
$$\geq(r_{ij}-r_{i'j'})\epsilon+\gamma\max\{\alpha,\beta\}\epsilon(r_{i'j'}-r_{ij})$$
$$=(1-\gamma\max\{\alpha,\beta\})\epsilon(r_{ij}-r_{i'j'})\geq 0,$$

where the first inequality is due to Lemma 4 and the last inequality is due to that $\alpha,\beta,\gamma\leq 1$ and $(i,j)\succeq(i',j')$. □

Proof of Theorem 2. We prove this theorem by induction on t. For $t=T+1$, it is obvious that the result holds. Suppose that the result holds for period $t+1$, $t\leq T$. We show that it also holds for period t.

Now consider period $t\leq T$. Without loss of generality, we can assume that both x_i and y_j are positive in period t. Otherwise if $x_i=0$ or $y_j=0$, the result clearly holds because the only feasible choice for q_{ij} is zero and thus $q^*_{ij}=0=\min\{x_i,y_j\}$.

Fix any $(\mathbf{x},\mathbf{y})>\mathbf{0}$. Suppose that in the optimal matching policy $\mathbf{Q}^*$, $q^*_{ij}<\min\{x_i,y_j\}$. It is sufficient to show that there exists $\epsilon>0$ such that an alternative matching plan $\bar{\mathbf{Q}}$, in which

$\bar{q}_{ij} = q^*_{ij} + \epsilon$, weakly dominates $\mathbf{Q}^*$. In other words, the firm can improve weakly by matching ϵ more of type i demand and type j supply.

One of the following scenarios must hold for the post-matching quantities u^*_i and v^*_j: Case (i) $u^*_i > 0$ and $v^*_j > 0$; Case (ii) $u^*_i = 0$ and $v^*_j > 0$, or $u^*_i > 0$ and $v^*_j = 0$; Case (iii) $u^*_i = 0$ and $v^*_j = 0$. The ideas of constructing a weakly dominating policy for cases (i) and (iii) are representative, which will be repetitively used later to prove the global priority structure (see Theorem 1).

Case (i): $u^*_i > 0$ and $v^*_j > 0$. We choose $\epsilon > 0$ such that $u^*_i - \epsilon > 0$ and $v^*_j - \epsilon > 0$. Consider an alternative matching plan $\bar{\mathbf{Q}} = \mathbf{Q}^* + \epsilon \mathbf{e}^{n\times m}_{ij}$, which is clearly feasible. Then,

$$\begin{aligned} &H_t(\bar{\mathbf{Q}}, \mathbf{x}, \mathbf{y}) - H_t(\mathbf{Q}^*, \mathbf{x}, \mathbf{y}) \\ =& r_{ij}\epsilon + (h+c)\epsilon + \gamma E V_{t+1}(\alpha \mathbf{u}^* + \mathbf{D} - \alpha\epsilon \mathbf{e}^n_i, \beta \mathbf{v}^* + \mathbf{S} - \beta\epsilon \mathbf{e}^m_j) - \gamma E V_{t+1}(\alpha \mathbf{u}^* + \mathbf{D}, \beta \mathbf{v}^* + \mathbf{S}). \end{aligned} \tag{8}$$

If $t = T$, then $H_t(\bar{\mathbf{Q}}, \mathbf{x}, \mathbf{y}) - H_t(\mathbf{Q}^*, \mathbf{x}, \mathbf{y}) = r_{ij}\epsilon + (h+c)\epsilon \geq 0$.

If $t < T$, then by the induction hypothesis, in period $t+1$, the optimal quantity to match between type i demand and type j supply is $q^*_{ij}(t+1) = \min\{x_i(t+1), y_j(t+1)\}$. Consider the case $\beta \geq \alpha$. It is easy to see that

$$V_{t+1}(\alpha \mathbf{u}^* + \mathbf{D} - \alpha\epsilon \mathbf{e}^n_i, \beta \mathbf{v}^* + \mathbf{S} - \beta\epsilon \mathbf{e}^m_j) = V_{t+1}(\alpha \mathbf{u}^* + \mathbf{D} + (\beta-\alpha)\epsilon \mathbf{e}^n_i, \beta \mathbf{v}^* + \mathbf{S}) - \beta\epsilon r_{ij}, \tag{9}$$

because of the greedy matching of pair (i, j) for the subsequent periods.

Now compare two systems that start in period $t+1$ with the states $(\alpha \mathbf{u}^* + \mathbf{D} + (\beta-\alpha)\epsilon \mathbf{e}^n_i, \beta \mathbf{v}^* + \mathbf{S})$ and $(\alpha \mathbf{u}^* + \mathbf{D}, \beta \mathbf{v}^* + \mathbf{S})$, respectively. The former system has the option of holding the additional amount $(\beta - \alpha)\epsilon$ of type i demand and mimicking the optimal matching policy of the latter system from period $t+1$ to period T. In this way, the former system incurs the extra cost $c(\beta-\alpha)\epsilon \sum_{\tau=0}^{T-t} \alpha^\tau \gamma^\tau \leq c(\beta-\alpha)\epsilon/(1-\alpha\gamma) \leq c\epsilon$. That is,

$$V_{t+1}(\alpha \mathbf{u}^* + \mathbf{D} + (\beta-\alpha)\epsilon \mathbf{e}^n_i, \beta \mathbf{v}^* + \mathbf{S}) \geq V_{t+1}(\alpha \mathbf{u}^* + \mathbf{D}, \beta \mathbf{v}^* + \mathbf{S}) - c\epsilon. \tag{10}$$

Then, combining (8), (9) and (10), we have

$$\begin{aligned} &H_t(\bar{\mathbf{Q}}, \mathbf{x}, \mathbf{y}) - H_t(\mathbf{Q}^*, \mathbf{x}, \mathbf{y}) \\ \geq& r_{ij}\epsilon + (h+c)\epsilon - \gamma\beta\epsilon r_{ij} + \gamma E V_{t+1}(\alpha \mathbf{u}^* + \mathbf{D} + (\beta-\alpha)\epsilon \mathbf{e}^n_i, \beta \mathbf{v}^* + \mathbf{S}) - \gamma E V_{t+1}(\alpha \mathbf{u}^* + \mathbf{D}, \beta \mathbf{v}^* + \mathbf{S}) \\ \geq& r_{ij}\epsilon + (h+c)\epsilon - \gamma\beta\epsilon r_{ij} - \gamma c\epsilon \geq 0, \end{aligned}$$

which demonstrates that $\bar{\mathbf{Q}}$ weakly dominates $\mathbf{Q}^*$. Similarly, we can reach the same conclusion if $\alpha > \beta$.

Case (ii): Suppose that $u_i^* > 0$ and $v_j^* = 0$. By Theorem 1, under the conditions that $(i,j) \succeq (i',j)$ for all $i' \in \mathcal{D}$, we know that $q_{i'j}^* = 0$ for any $i' \in \mathcal{D}$ and $i' \neq i$. Then, $0 = v_j^* = y_j - \sum_{i'=1}^n q_{i'j}^* = y_j - q_{ij}^*$. Thus, $q_{ij}^* = y_j \geq \min\{x_i, y_j\}$, implying that $q_{ij}^* = \min\{x_i, y_j\}$ because $q_{ij}^* \leq \min\{x_i, y_j\}$.

Similarly, we can prove $q_{ij}^* = \min\{x_i, y_j\}$, if $u_i^* = 0$ and $v_j^* > 0$.

Case (iii): $u_i^* = v_j^* = 0$. Assume $q_{ij}^* < \min\{x_i, y_j\}$. Then, there must exist $j' \neq j$ and $i' \neq i$ such that $q_{ij'}^* > 0$ and $q_{i'j}^* > 0$. We choose $\epsilon > 0$ such that $q_{i'j}^* - \epsilon > 0$ and $q_{ij'}^* - \epsilon > 0$ and define $\bar{\mathbf{Q}}$ as $\tilde{\mathbf{Q}} = \mathbf{Q}^* + \epsilon(\mathbf{e}_{ij}^{n\times m} + \mathbf{e}_{i'j'}^{n\times m} - \mathbf{e}_{i'j}^{n\times m} - \mathbf{e}_{ij'}^{n\times m})$. The decision $\bar{\mathbf{Q}}$ is feasible, because $\bar{\mathbf{Q}} \geq \mathbf{0}$ and the post-matching levels of $\bar{\mathbf{Q}}$ are the same as that of $\mathbf{Q}^*$. Then, $H_t(\bar{\mathbf{Q}}, \mathbf{x}, \mathbf{y}) - H_t(\mathbf{Q}^*, \mathbf{x}, \mathbf{y}) = \epsilon(r_{ij} + r_{i'j'} - r_{i'j} - r_{ij'}) \geq 0$, implying that $\bar{\mathbf{Q}}$, in which $\bar{q}_{ij} = q_{ij}^* + \epsilon$, weakly dominates $\mathbf{Q}^*$. Following the same argument, we can always find an optimal decision $\bar{\mathbf{Q}}$ in which $\bar{q}_{ij} = \min\{x_i, y_j\}$. □

Proof of Theorem 3. (i) Without loss of generality, suppose that $r_{ij} = f(d_{ij}) = r_0 - d_{ij}$. The condition $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i,j')}$ implies that $d_{ij'} = d_{ij} + \vec{d}(j',j)$. Then, $r_{ij} - r_{ij'} = \vec{d}(j',j)$.

For any $i' \in \mathcal{D}$, $r_{i'j} - r_{i'j'} = d_{i'j'} - d_{i'j}$. To verify condition (D), it suffices to show that $r_{i'j} - r_{i'j'} \leq r_{ij} - r_{ij'}$, which is equivalent to $d_{i'j'} \leq d_{i'j} + \vec{d}(j',j)$ following the above arguments. The latter inequality is simply the triangular inequality. The other half can be proved analogously.

(ii) In the case of undirected line segment, if $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i',j')}$ and $\overleftarrow{(i,j)}$ has the same direction with $\overleftarrow{(i',j')}$, we have $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i,j')} \subseteq \overleftarrow{(i',j')}$. By part (i), $(i,j) \succeq (i,j') \succeq (i',j')$.

(iii) If $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i',j')}$, since $\overleftarrow{(i,j)}$ has the same direction with $\overleftarrow{(i',j')}$ is automatically satisfied for the directed line segment and directed circle, by part (ii), we have $(i,j) \succeq (i',j')$.

It remains to show that $(i,j) \succeq (i',j')$ implies $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i',j')}$. Suppose $(i,j) \succeq (i',j')$. We can assume without loss of generality that $(i,j) = (i_1,j_1) \succeq (i_1,j_2) \succeq (i_2,j_2) \succeq \cdots \succeq (i_\ell,j_\ell) = (i',j')$. By definition of the partial order, $(i_1,j_1) \succeq (i_1,j_2)$ implies $r_{i_1j_1} \geq r_{i_1j_2}$, thus $d_{i_1j_1} \leq d_{i_1j_2}$. In either the directed line segment case or the directed circle case, this suggests $\overleftarrow{(i_1,j_1)} \subseteq \overleftarrow{(i_1,j_2)}$. Repeating this argument we get $\overleftarrow{(i_1,j_1)} \subseteq \overleftarrow{(i_1,j_2)} \subseteq \overleftarrow{(i_2,j_2)} \subseteq \cdots \subseteq \overleftarrow{(i_l,j_l)}$. Thus $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i',j')}$. □

Proof of Proposition 2. First, we show that $(i,j) \succeq (i',j)$ if $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i',j)}$. If $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i',j)}$, then i' must be closer to the end point e, which leads to $r_i^a \geq r_{i'}^a$. Thus $r_{ij} = f(d_{ij}) + r_i^a \geq f(d_{i'j}) + r_{i'}^a = r_{i'j}$, given that $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i',j)}$. To verify Condition (D), we can use the fact that $r_{ij} + r_{i'j'} = f(d_{ij}) + f(d_{i'j'}) + r_i^a + r_{i'}^a \geq f(d_{ij'}) + f(d_{i'j}) + r_i^a + r_{i'}^a$, where the inequality holds because $f(d_{ij}) + f(d_{i'j'}) \geq f(d_{ij'}) + f(d_{i'j})$ that has been already proved in Theorem 3 part (i).

Then, following the same analysis as that of Theorem 3 part (iii), we can also show that $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i',j')}$ if $(i,j) \succeq (i',j')$. □

Proof of Theorem 4. If $\overleftarrow{(i,j)}$ does not contain any types other than i and j themselves, then $\overleftarrow{(i,j)} \subseteq \overleftarrow{(i',j')}$ for any compatible pair (i',j') (in the case of directed line segment, we require $j' \to i'$;

otherwise the arc (i',j') is “incompatible”). By Theorem 3 part (iii), $(i,j) \succeq (i',j')$. In particular, $(i,j) \succeq (i,j')$ and $(i,j) \succeq (i',j)$ for any $i' \in \mathcal{D}$ and $j' \in \mathcal{S}$. By Theorem 2, $i \in \mathcal{D}$ and $j \in \mathcal{S}$ should be matched with each other as much as possible. $\square$

Proof of Theorem 5. By Observation 1 and Theorem 4, it is optimal to match the perfect pair as much as possible; that is done in the first round. Next we consider after round 1 how to match the imperfect pairs $(1,2)$ and $(2,1)$. If $x_1 \geq y_1$ and $x_2 \geq y_2$ or $x_1 \leq y_1$ and $x_2 \leq y_2$, it is obvious that $q^*_{12} = q^*_{21} = 0$.

Consider the case where $x_1 > y_1$ and $x_2 < y_2$. (The same argument applies to the case where $x_1 < y_1$ and $x_2 > y_2$.) After round 1, the remaining quantities for type 1 demand and type 2 supply is $x_1 - y_1 > 0$ and $y_2 - x_2 > 0$ respectively. There is no remaining unmatched type 2 demand and type 1 supply, and thus $q^*_{21} = 0$. It remains to determine the optimal matching quantity q^*_{12}, which is equivalent to determining some optimal protection level $a^*_1(t,\mathbf{x},\mathbf{y})$. To see this, note that η^+ and η^- would be the unmatched type 1 demand and type 2 supply remaining after the imperfect pair $(1,2)$ has been matched as much as possible, where $\eta = \eta_1 + \eta_2 = (x_1 - y_1) - (y_2 - x_2)$. When the protection level is a_1, the post-matching levels of type 1 demand and type 2 supply are $u_1 = \eta^+ + a_1$ and $v_2 = \eta^- + a_1$, respectively. The protection level needs to satisfy the nonnegativity constraint $a_1 \geq 0$ and ensure the matching quantity $q_{12} = \eta_1 - u_1 = \eta_1 - \eta^+ - a_1 \geq 0$, resulting in $a_1 \leq \eta_1 - \eta^+$. After Round 1, the cost-to-go function can be written in terms of the protection level a_1 as:

$$\begin{aligned}\widetilde{H}_t(a_1,\mathbf{x},\mathbf{y}) =& r_{11}y_1 + r_{22}x_2 + r_{12}(\eta_1 - \eta^+ - a_1) - c(\eta^+ + a_1) - h(\eta^- + a_1) \\ &+ \gamma E V_{t+1}\left(\alpha(\eta^+ + a_1) + D_1, D_2, S_1, \beta(\eta^- + a_1) + S_2\right) \\ =& r_{11}y_1 + r_{22}x_2 + r_{12}(\eta_1 - \eta^+) - c\eta^+ - h\eta^- + \hat{H}_t(a_1,\eta),\end{aligned}$$

where

$$\hat{H}_t(a_1,\eta) = -(r_{12} + c + h)a_1 + \gamma E V_{t+1}\left(\alpha(\eta^+ + a_1) + D_1, D_2, S_1, \beta(\eta^- + a_1) + S_2\right) \tag{11}$$

depends on $(\mathbf{x},\mathbf{y})$ only through η. The optimal protection level can be obtained by solving $\max_{0 \leq a_1 \leq \eta_1 - \eta^+} \hat{H}_t(a_1,\eta)$. As with the proof of Proposition 1, it is easy to show that $\hat{H}_t(a_1,\eta)$ is concave in a_1. Thus, the optimal protection level $a^*_1(t,\mathbf{x},\mathbf{y}) = a^*_1(t,\eta) = \min\{\bar{a}_1(t,\eta), \eta_1 - \eta^+\}$, where $\bar{a}_1(t,\eta) \in \arg\max_{a_1 \geq 0} \hat{H}_t(a_1,\eta)$. $\square$

Proof of Corollary 2. Consider the case in which $x_1 \geq y_1$ and $x_2 \leq y_2$. If $\alpha = 0$, in the proof of Theorem 5, (11) reduces to $\hat{H}_t(a_1,\eta) = -(r_{12} + c + h)a_1 + \gamma E V_{t+1}\left(D_1, D_2, S_1, \beta(\eta^- + a_1) + S_2\right)$. To optimize the protection level a_1 is equivalent to optimizing the post-matching level $v_2 = \eta^- + a_1$ of supply type 2. Let $\hat{v}_2(t) \in \arg\max_{v_2 \geq 0} -(r_{12} + c + h)v_2 + \gamma E V_{t+1}(D_1, D_2, S_1, \beta v_2 + S_2)$, which is independent of η. Since $0 \le a_1 \le \eta_1 - \eta^+$, $\eta^- \le v_2 = \eta^- + a_1 \le \eta_2^-$. Then, $v_2^* = \max\{\hat{v}_2(t) \wedge \eta_2^-, \eta^-\}$, $a_1^*(t,\eta) = v_2^* - \eta^- = [\hat{v}_2(t) \wedge \eta_2^- - \eta^-]^+$ and $q_{12}^* = \eta_2^- - v_2^* = \eta_2^- - \max\{\hat{v}_2(t) \wedge \eta_2^-, \eta^-\}$. Analogously, we can show the desired result for $\beta = 0$. $\square$

Proof of Corollary 3. From Theorem 1, the arc (i,j) has priority over (i,j') and (i',j) for all $j' > j$ and $i' > i$, which immediately leads to the result. $\square$

Derivation of DP (2). Given a state $(\mathbf{x},\mathbf{y})$ and the total matching quantity Q in period t, the unmatched amount of type i' demand is $(\tilde{x}_{i'} - Q)^+ - (\tilde{x}_{i'-1} - Q)^+$, where the first term is the unmatched amount of demand in types $1,\dots,i'$, and the second term is the unmatched amount of demand in types $1,\dots,i'-1$. Thus a total amount of $x_{i'} - [(\tilde{x}_{i'} - Q)^+ - (\tilde{x}_{i'-1} - Q)^+]$ of type i' demand is matched with some supply in period t. Similarly, an amount $y_{j'} - [(\tilde{y}_{j'} - Q)^+ - (\tilde{y}_{j'-1} - Q)^+]$ of type j' supply is matched with some demand in period t. Thus the total reward from matching in period t is

$$\begin{aligned}
&\sum_{i'=1}^{n} r_{i'}^d \left\{x_{i'} - [(\tilde{x}_{i'} - Q)^+ - (\tilde{x}_{i'-1} - Q)^+]\right\} + \sum_{j'=1}^{m} r_{j'}^s \left\{y_{j'} - [(\tilde{y}_{j'} - Q)^+ - (\tilde{y}_{j'-1} - Q)^+]\right\} \\
=& \sum_{i'=1}^{n} r_{i'}^d x_{i'} + \sum_{j'=1}^{m} r_{j'}^s y_{j'} - \sum_{i'=1}^{n} \left(r_{i'}^d - r_{i'+1}^d\right)(\tilde{x}_{i'} - Q)^+ - \sum_{j'=1}^{m} \left(r_{j'}^s - r_{j'+1}^s\right)(\tilde{y}_{j'} - Q)^+,
\end{aligned}$$

where $r_{n+1}^d = r_{m+1}^s \equiv 0$. $\square$

Proof of Lemma 5. It is easy to see that G_t is concave in Q within the interior of the ranges $\tilde{x}_{i-1} \le Q < \tilde{x}_i$ and $\tilde{y}_{j-1} \le Q < \tilde{y}_j$.

Without loss of generality, we assume that $\tilde{x}_i \in (\tilde{y}_{j-1}, \tilde{y}_j)$. We show that G_t is concave in the neighborhood of a breakpoint $a = \tilde{x}_i$. To this end, it suffices to show that $G_t(a+\epsilon,\mathbf{x},\mathbf{y}) - G_t(a,\mathbf{x},\mathbf{y}) \le G_t(a,\mathbf{x},\mathbf{y}) - G_t(a-\epsilon,\mathbf{x},\mathbf{y})$, where $0 < \epsilon < \min\{\tilde{x}_i - \tilde{y}_{j-1}, \tilde{y}_j - \tilde{x}_i\}$.

One the one hand, we have

$$\begin{aligned}
& G_t(a,\mathbf{x},\mathbf{y}) - G_t(a-\epsilon,\mathbf{x},\mathbf{y}) \\
=& (r_i^d + r_j^s + c + h)\epsilon \\
& + \gamma E V_{t+1}(\mathbf{D}_{[1,i]}, \alpha \mathbf{x}_{[i+1,n]} + \mathbf{D}_{[i+1,n]}, \mathbf{S}_{[1,j-1]}, \beta(\tilde{y}_j - \tilde{x}_i) + S_j, \beta \mathbf{y}_{[j+1,m]} + \mathbf{S}_{[j+1,m]}) \\
& - \gamma E V_{t+1}(\mathbf{D}_{[1,i-1]}, \alpha\epsilon + D_i, \alpha \mathbf{x}_{[i+1,n]} + \mathbf{D}_{[i+1,n]}, \mathbf{S}_{[1,j-1]}, \beta(\tilde{y}_j - \tilde{x}_i) + \beta\epsilon + S_j, \beta \mathbf{y}_{[j+1,m]} + \mathbf{S}_{[j+1,m]}) \\
\ge& (r_i^d + r_j^s + c + h)\epsilon - \gamma\alpha(r_i^d - r_{i+1}^d)\epsilon \\
& + \gamma E V_{t+1}(\mathbf{D}_{[1,i]}, \alpha \mathbf{x}_{[i+1,n]} + \mathbf{D}_{[i+1,n]}, \mathbf{S}_{[1,j-1]}, \beta(\tilde{y}_j - \tilde{x}_i) + S_j, \beta \mathbf{y}_{[j+1,m]} + \mathbf{S}_{[j+1,m]}) \\
& - \gamma E V_{t+1}(\mathbf{D}_{[1,i]}, \alpha x_{i+1} + D_{i+1} + \alpha\epsilon, \alpha \mathbf{x}_{[i+2,n]} + \mathbf{D}_{[i+2,n]}, \mathbf{S}_{[1,j-1]}, \beta(\tilde{y}_j - \tilde{x}_i) + \beta\epsilon + S_j, \beta \mathbf{y}_{[j+1,m]} + \mathbf{S}_{[j+1,m]}),
\end{aligned}$$

where the inequality follows from Lemma 3 (set ϵ_t^2 to zero) and the fact that $-\sum_{j'=1}^m \lambda_{j'}(r_{ij'} - r_{i'j'}) = -\sum_{j'=1}^m \lambda_{j'}(r_i^d - r_{i'}^d) \geq -(r_i^d - r_{i'}^d)\alpha\epsilon$ if $\sum_{j'=1}^m \lambda_{j'} \leq \alpha\epsilon$. On the other hand, we have

$$\begin{aligned}
&G_t(a+\epsilon, \mathbf{x}, \mathbf{y}) - G_t(a, \mathbf{x}, \mathbf{y}) \\
=&(r_{i+1}^d + r_j^s + c + h)\epsilon \\
&+ \gamma E V_{t+1}(\mathbf{D}_{[1,i]}, \alpha x_{i+1} - \alpha\epsilon + D_{i+1}, \alpha \mathbf{x}_{[i+2,n]} + \mathbf{D}_{[i+2,n]}, \mathbf{S}_{[1,j-1]}, \beta(\tilde{y}_j - \tilde{x}_i) - \beta\epsilon + S_j, \beta\mathbf{y}_{[j+1,m]} + \mathbf{S}_{[j+1,m]}) \\
&- \gamma E V_{t+1}(\mathbf{D}_{[1,i]}, \alpha x_{i+1} + D_{i+1}, \alpha \mathbf{x}_{[i+2,n]} + \mathbf{D}_{[i+2,n]}, \mathbf{S}_{[1,j-1]}, \beta(\tilde{y}_j - \tilde{x}_i) + S_j, \beta\mathbf{y}_{[j+1,m]} + \mathbf{S}_{[j+1,m]}).
\end{aligned}$$

Since $\gamma, \alpha \in [0,1]$ and $r_i^d \geq r_{i+1}^d$, $(r_i^d + r_j^s + c + h)\epsilon - \gamma\alpha(r_i^d - r_{i+1}^d)\epsilon \geq (r_i^d + r_j^s + c + h)\epsilon - (r_i^d - r_{i+1}^d)\epsilon = (r_{i+1}^d + r_j^s + c + h)\epsilon$. Then, by the concavity of $V_{t+1}(\cdot)$ (see Proposition 1), it follows that $G_t(a+\epsilon, \mathbf{x}, \mathbf{y}) - G_t(a, \mathbf{x}, \mathbf{y}) \leq G_t(a, \mathbf{x}, \mathbf{y}) - G_t(a-\epsilon, \mathbf{x}, \mathbf{y})$. □

Proof of Theorem 6. Take the dynamic view of the top-down matching procedure and consider the scenario when it gets to the matching of type i demand and type j supply. The available amount of type i demand is $\overline{x}_i \stackrel{\text{def}}{=} x_i - (\tilde{y}_{j-1} - \tilde{x}_{i-1})^+$ and that of type j supply is $\overline{y}_j \stackrel{\text{def}}{=} y_j - (\tilde{x}_{i-1} - \tilde{y}_{j-1})^+$. If type i demand and type j supply were matched as much as possible, after the matching the amount of type i demand would become $\underline{x}_i \stackrel{\text{def}}{=} (\tilde{x}_i - \tilde{y}_j)^+$ and that of type j supply would become $\underline{y}_j \stackrel{\text{def}}{=} (\tilde{y}_j - \tilde{x}_i)^+$. Note that we have $\overline{x}_i - \underline{x}_i = x_i - (\tilde{y}_{j-1} - \tilde{x}_{i-1})^+ - (\tilde{x}_i - \tilde{y}_j)^+ = y_j - (\tilde{y}_{j-1} - \tilde{x}_{i-1})^- - (\tilde{x}_i - \tilde{y}_j)^- = y_j - (\tilde{x}_{i-1} - \tilde{y}_{j-1})^+ - (\tilde{y}_j - \tilde{x}_i)^+ = \overline{y}_j - \underline{y}_j$, where the second equality is due to $x_i - y_j = (\tilde{y}_{j-1} - \tilde{x}_{i-1}) + (\tilde{x}_i - \tilde{y}_j)$ and $z = z^+ - z^-$, and the third equality is due to $(-z)^- = z^+$. Thus, determining the optimal matching quantity between type i demand and type j supply is equivalent to finding the optimal protection level $a_{ij}^*(t)$ such that the post-matching levels $u_i^* = \underline{x}_i + a_{ij}^*(t) = (\tilde{x}_i - \tilde{y}_j)^+ + a_{ij}^*(t)$ and $v_j^* = \underline{y}_j + a_{ij}^*(t) = (\tilde{y}_j - \tilde{x}_i)^+ + a_{ij}^*(t)$.

Let $a_{ij}^*(t, \tilde{x}_i - \tilde{y}_j, \mathbf{x}_{[i+1,n]}, \mathbf{y}_{[j+1,m]}) \in \arg\max_{a \geq 0} \big[-(r_i^d + r_j^s + c + h)a + \gamma E V_{t+1}(\mathbf{D}_{[1,i-1]}, \alpha(\underline{x}_i + a) + D_i, \alpha\mathbf{x}_{[i+1,n]} + \mathbf{D}_{[i+1,n]}, \mathbf{S}_{[1,j-1]}, \beta(\underline{y}_j + a) + S_j, \beta\mathbf{y}_{[j+1,m]} + \mathbf{S}_{[j+1,m]})\big]$.

If $\overline{x}_i - \underline{x}_i = x_i - (\tilde{y}_{j-1} - \tilde{x}_{i-1})^+ - (\tilde{x}_i - \tilde{y}_j)^+ > a_{ij}^*(t)$, then it is feasible and optimal to match type i demand with type j supply until the quantity of type i demand reduces to $\underline{x}_i + a_{ij}^*(t) = (\tilde{x}_i - \tilde{y}_j)^+ + a_{ij}^*(t)$. Otherwise, it is optimal not to match type i demand with type j supply. □

Proof of Lemma 6. By Lemma 3 (set ϵ_t^2 to zero), there exists $(\lambda_1, \ldots, \lambda_m) \geq 0$ such that $\sum_{j'=1}^m \lambda_{j'} \leq \epsilon$ and $V_t(\mathbf{x}, \mathbf{y}) - V_t(\mathbf{x} + \epsilon\mathbf{e}_i^n - \epsilon\mathbf{e}_{i'}^n, \mathbf{y}) = V_t((\mathbf{x} + \epsilon\mathbf{e}_i^n - \epsilon\mathbf{e}_{i'}^n) - \epsilon\mathbf{e}_i^n + \epsilon\mathbf{e}_{i'}^n, \mathbf{y}) - V_t(\mathbf{x} + \epsilon\mathbf{e}_i^n - \epsilon\mathbf{e}_{i'}^n, \mathbf{y}) \geq -\sum_{j'=1}^m \lambda_{j'}(r_{ij'} - r_{i'j'}) = -\sum_{j'=1}^m \lambda_{j'}(r_i^d - r_{i'}^d)$. If $i < i'$, then $r_i^d > r_{i'}^d$ and $V_t(\mathbf{x}, \mathbf{y}) - V_t(\mathbf{x} + \epsilon\mathbf{e}_i^n - \epsilon\mathbf{e}_{i'}^n, \mathbf{y}) \geq -\sum_{j'=1}^m \lambda_{j'}(r_i^d - r_{i'}^d) \geq -(r_i^d - r_{i'}^d)\epsilon$. Then, for $1 \leq k < n$, we have

$$\begin{aligned}
\tilde{V}_t(\tilde{\mathbf{x}} + \epsilon\mathbf{e}_k^n, \tilde{\mathbf{y}}) - \tilde{V}_t(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) =& V_t((\tilde{\mathbf{x}} + \epsilon\mathbf{e}_k^n)\mathbf{U}_n^{-1}, \tilde{\mathbf{y}}\mathbf{U}_m^{-1}) - V_t(\tilde{\mathbf{x}}\mathbf{U}_n^{-1}, \tilde{\mathbf{y}}\mathbf{U}_m^{-1}) - (\tilde{\mathbf{x}} + \epsilon\mathbf{e}_k^n)\mathbf{U}_n^{-1}(\mathbf{r}^d)^{\mathsf{T}} - \tilde{\mathbf{x}}\mathbf{U}_n^{-1}(\mathbf{r}^d)^{\mathsf{T}} \\
=& V_t(\mathbf{x} + \epsilon\mathbf{e}_k^n - \epsilon\mathbf{e}_{k+1}^n, \mathbf{y}) - V_t(\mathbf{x}, \mathbf{y}) - (r_k^d - r_{k+1}^d)\epsilon \leq 0.
\end{aligned}$$

Thus, we have shown that $\tilde{V}_t(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ is decreasing in $\tilde{x}_k$ for $1 \le k < n$. Similarly, we can show that $\tilde{V}_t(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ is decreasing in $\tilde{y}_k$ for $1 \le k < m$. $\square$

Proof of Lemma 7. The proof is by induction on t. Clearly, $\tilde{V}_{T+1}(\tilde{\mathbf{x}},\tilde{\mathbf{y}}) \equiv 0$ is $L^\natural$-concave in $(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$. We suppose that $\tilde{V}_{t+1}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ is $L^\natural$-concave in $(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$. Then by definition of $L^\natural$-concavity and submodularity, for any given $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{S}}$, $\tilde{V}_{t+1}(\alpha\tilde{\mathbf{x}}+\tilde{\mathbf{D}},\alpha\tilde{\mathbf{y}}+\tilde{\mathbf{S}})$ is $L^\natural$-concave in $(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$. Now consider period t. Since $Q \le \min\{\tilde{x}_n,\tilde{y}_m\}$ and $\alpha = \beta$,

$$\begin{aligned}&\tilde{V}_{t+1}(\alpha(\tilde{\mathbf{x}}-Q\mathbf{1}^n)^+ + \tilde{\mathbf{D}}, \alpha(\tilde{\mathbf{y}}-Q\mathbf{1}^m)^+ + \tilde{\mathbf{S}})\\ &= \tilde{V}_{t+1}(\alpha(\tilde{\mathbf{x}}_{[1,n-1]} - Q\mathbf{1}^{n-1})^+ + \tilde{\mathbf{D}}_{[1,n-1]}, \alpha(\tilde{x}_n - Q) + \tilde{D}_n, \alpha(\tilde{\mathbf{y}}_{[1,m-1]} - Q\mathbf{1}^{m-1})^+ + \tilde{\mathbf{S}}_{[1,m-1]}, \alpha(\tilde{y}_m - Q) + \tilde{S}_m),\end{aligned}$$

which is $L^\natural$-concave in $(Q,\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ by applying Chen et al. (2014, Lemma 4) and noting the monotonicity proved in Lemma 6. By Simchi-Levi et al. (2014, Proposition 2.3.4(c)), $E_{\tilde{\mathbf{D}},\tilde{\mathbf{S}}}[\tilde{V}_{t+1}(\alpha(\tilde{\mathbf{x}} - Q\mathbf{1}^n)^+ + \tilde{\mathbf{D}}, \alpha(\tilde{\mathbf{y}} - Q\mathbf{1}^m)^+ + \tilde{\mathbf{S}})]$ is $L^\natural$-concave in $(Q,\tilde{\mathbf{x}},\tilde{\mathbf{y}})$, thus the last term in (3) is $L^\natural$-concave in $(Q,\tilde{\mathbf{x}},\tilde{\mathbf{y}})$. The first two terms in (3) are $L^\natural$-concave in $(Q,\tilde{\mathbf{x}},\tilde{\mathbf{y}})$, because $-(\tilde{x}_{i'} - Q)^+$ is supermodular in $(Q,\tilde{x}_{i'})$, $-(\tilde{y}_{j'} - Q)^+$ is supermodular in $(Q,\tilde{y}_{j'})$ and $L^\natural$-concavity is preserved under any nonnegative linear combination. Since the other terms are linear, $\tilde{G}_t(Q,\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ is $L^\natural$-concave in $(Q,\tilde{\mathbf{x}},\tilde{\mathbf{y}})$. By Simchi-Levi et al. (2014, Proposition 2.3.4(e)), $\tilde{V}_t(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ is $L^\natural$-concave in $(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$. This completes the induction. $\square$

Proof of Theorem 7. Monotonicity of $Q_t^*(\mathbf{x},\mathbf{y})$. Since $L^\natural$-concavity implies supermodularity, by Lemma 7, $\tilde{G}_t(Q,\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ is $L^\natural$-concave, a fortiori, supermodular in $(Q,\tilde{\mathbf{x}},\tilde{\mathbf{y}})$. By Simchi-Levi et al. (2014, Theorem 2.2.8), the optimal solution to (3), denoted by $\hat{Q}_t(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$, is nondecreasing in $(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$. Since the higher the original state $(\mathbf{x},\mathbf{y})$, the higher the transformed state $(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$, the optimal solution $Q_t^*(\mathbf{x},\mathbf{y})$, expressed in terms of the original state, is nondecreasing in $(\mathbf{x},\mathbf{y})$.

(i) Since $(\mathbf{x}+\epsilon\mathbf{e}_1^n, \mathbf{y}+\epsilon\mathbf{e}_1^m)$ is a state that has ϵ more type 1 demand and type 1 supply than state $(\mathbf{x},\mathbf{y})$, it is optimal to match type 1 demand and supply as much as possible; thus after the first round of matching type 1 demand and supply, there are the same levels of the remaining types for the system with state $(\mathbf{x}+\epsilon\mathbf{e}_1^n, \mathbf{y}+\epsilon\mathbf{e}_1^m)$ and with state $(\mathbf{x},\mathbf{y})$. Thus the optimal matching decisions for the remaining types must be the same for the two states, and as a result, $Q_t^*(\mathbf{x}+\epsilon\mathbf{e}_1^n, \mathbf{y}+\epsilon\mathbf{e}_1^m) = Q_t^*(\mathbf{x},\mathbf{y}) + \epsilon$.

(ii) By the definition of $L^\natural$-concavity, $\tilde{G}_t(Q-\xi, \tilde{\mathbf{x}}-\xi\mathbf{1}^n, \tilde{\mathbf{y}}-\xi\mathbf{1}^m)$ is supermodular in $(Q,\tilde{\mathbf{x}},\tilde{\mathbf{y}},\xi)$. Then, for $Q > \hat{Q}_t(\tilde{\mathbf{x}},\tilde{\mathbf{y}}) + \epsilon$, we have

$$\tilde{G}_t(Q,\tilde{\mathbf{x}}+\epsilon\mathbf{1}^n,\tilde{\mathbf{y}}+\epsilon\mathbf{1}^m) - \tilde{G}_t(\hat{Q}_t(\tilde{\mathbf{x}},\tilde{\mathbf{y}})+\epsilon, \tilde{\mathbf{x}}+\epsilon\mathbf{1}^n, \tilde{\mathbf{y}}+\epsilon\mathbf{1}^m) \le \tilde{G}_t(Q-\epsilon,\tilde{\mathbf{x}},\tilde{\mathbf{y}}) - \tilde{G}_t(\hat{Q}_t(\tilde{\mathbf{x}},\tilde{\mathbf{y}}),\tilde{\mathbf{x}},\tilde{\mathbf{y}}) \le 0,$$

where the first inequality is derived by definition of supermodularity and the second inequality is due to the optimality of $\hat{Q}_t$. This implies that any matching quantity $Q > \hat{Q}_t(\tilde{\mathbf{x}},\tilde{\mathbf{y}}) + \epsilon$ is no better than $\hat{Q}_t(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) + \epsilon$ for the state $(\tilde{\mathbf{x}} + \epsilon \mathbf{1}^n, \tilde{\mathbf{y}} + \epsilon \mathbf{1}^m)$. Therefore, $\hat{Q}_t(\tilde{\mathbf{x}} + \epsilon \mathbf{1}^n, \tilde{\mathbf{y}} + \epsilon \mathbf{1}^m) \le \hat{Q}_t(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) + \epsilon$. By the monotonicity of $\hat{Q}_t(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$, $\hat{Q}_t(\tilde{\mathbf{x}} + \epsilon \mathbf{1}^n, \tilde{\mathbf{y}}) \le \hat{Q}_t(\tilde{\mathbf{x}} + \epsilon \mathbf{1}^n, \tilde{\mathbf{y}} + \epsilon \mathbf{1}^m) \le \hat{Q}_t(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) + \epsilon$. Expressed in the original state, $Q_t^*(\mathbf{x} + \epsilon \mathbf{e}_1^n, \mathbf{y}) \le Q_t^*(\mathbf{x}, \mathbf{y}) + \epsilon$, which proves the last inequality in part (ii).

For any two original states $(\mathbf{x} + \epsilon \mathbf{e}_k^n, \mathbf{y})$ and $(\mathbf{x} + \epsilon \mathbf{e}_{k+1}^n, \mathbf{y})$, $k = 1, \ldots, n-1$, their transformed states can be ordered as $(\tilde{\mathbf{x}} + \epsilon \mathbf{1}_{[k,n]}, \tilde{\mathbf{y}}) \ge (\tilde{\mathbf{x}} + \epsilon \mathbf{1}_{[k+1,n]}, \tilde{\mathbf{y}})$, where $\mathbf{1}_{[k,n]}$ is an n-dimensional vector with the k-th up to n-th entry being one and the rest of the entries being all zeros. By the monotonicity of $\hat{Q}_t(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$, $\hat{Q}_t(\tilde{\mathbf{x}} + \epsilon \mathbf{1}_{[k,n]}, \tilde{\mathbf{y}}) \ge \hat{Q}_t(\tilde{\mathbf{x}} + \epsilon \mathbf{1}_{[k+1,n]}, \tilde{\mathbf{y}})$. Translated into the original state, $Q_t^*(\mathbf{x} + \epsilon \mathbf{e}_k^n, \mathbf{y}) \ge Q_t^*(\mathbf{x} + \epsilon \mathbf{e}_{k+1}^n, \mathbf{y})$ and thus, $Q_t^*(\mathbf{x} + \epsilon \mathbf{e}_{k+1}^n, \mathbf{y}) - Q_t^*(\mathbf{x}, \mathbf{y}) \le Q_t^*(\mathbf{x} + \epsilon \mathbf{e}_k^n, \mathbf{y}) - Q_t^*(\mathbf{x}, \mathbf{y})$. Combining that with $Q_t^*(\mathbf{x} + \epsilon \mathbf{e}_1^n, \mathbf{y}) \le Q_t^*(\mathbf{x}, \mathbf{y}) + \epsilon$, we have the desired series of inequalities in part (ii), with the first inequality implied by the monotonicity of $Q_t^*(\mathbf{x}, \mathbf{y})$.

(iii) The series of inequalities can be proved analogously to part (ii). □

Proof of Corollary 4. In view of Theorem 6, the protection level $a_{ij}^*(t)$ plays a role in deciding how much to match type i demand and type j supply, which happens only when $\tilde{x}_i > \tilde{y}_{j-1}$ and $\tilde{x}_{i-1} < \tilde{y}_j$ and types $1, 2, \ldots, i-1$ demand and types $1, 2, \ldots, j-1$ supply have been fully matched. The optimal *total* quantity to be matched between demand types $1, \ldots, i$ and supply types $1, \ldots, j$ is given by $q_{ij}^* \in \arg\max_{0 \le q \le \min\{\tilde{x}_i, \tilde{y}_j\}} \tilde{G}_t(q, \tilde{\mathbf{x}}, \tilde{\mathbf{y}})$. By the concavity of $\tilde{G}_t(q, \tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ with respect to q, $q_{ij}^* = \min\{q^*, \tilde{x}_i, \tilde{y}_j\}$ where $q^* \in \arg\max_{q \ge 0} \tilde{G}_t(q, \tilde{\mathbf{x}}, \tilde{\mathbf{y}})$. Since $q^* \ge \max\{\tilde{x}_{i-1}, \tilde{y}_{j-1}\}$ (because types $1, 2, \ldots, i-1$ demand and types $1, 2, \ldots, j-1$ supply have been fully matched) and $\tilde{x}_i > \tilde{y}_{j-1}$ and $\tilde{x}_{i-1} < \tilde{y}_j$, we have $q_{ij}^* = \min\{q^*, \tilde{x}_i, \tilde{y}_j\} \ge \max\{\tilde{x}_{i-1}, \tilde{y}_{j-1}\}$. Thus, from (3), we see that q_{ij}^* depends only on $(\tilde{\mathbf{x}}_{[i,n]}, \tilde{\mathbf{y}}_{[j,m]})$.

Without loss of generality, consider the case $\tilde{x}_i \ge \tilde{y}_j$. In this case, given the protection level $a_{ij}^*(t)$, the quantity of type i demand after matching with type j supply can be expressed as $\tilde{x}_i - \tilde{y}_j + a_{ij}^*(t)$. On the other hand, it can also be expressed as $\tilde{x}_i - q_{ij}^*$. Equating these two expressions leads to $a_{ij}^*(t) = \tilde{y}_j - q_{ij}^*$. By $L^\natural$-concavity of $\tilde{G}_t$, analogous to the proof of Theorem 7, we can show that q_{ij}^* is nondecreasing in $\tilde{x}_{i'}$ and $\tilde{y}_{j'}$ for $i \le i' \le n$ and $j \le j' \le m$ with the increasing rate less than or equal to 1. Since $a_{ij}^*(t)$ depends only on $\eta_{ij} \equiv \tilde{x}_i - \tilde{y}_j$, $\tilde{x}_{[i+1,n]}$ and $\tilde{y}_{[j+1,m]}$, it is nonincreasing in η_{ij}, $\tilde{x}_{[i+1,n]}$ and $\tilde{y}_{[j+1,m]}$ and the decreasing rates are dominated by 1. Moreover, analogous to the proof of Theorem 7, in terms of the original state $(\mathbf{x}, \mathbf{y})$, for $i + 1 \le i' \le n-1$, $-1 \le a_{ij}^*(t, \eta_{ij} + \epsilon, \mathbf{x}_{[i+1,n]}, \mathbf{y}_{[j+1,m]}) - a_{ij}^*(t, \eta_{ij}, \mathbf{x}_{[i+1,n]}, \mathbf{y}_{[j+1,m]}) \le a_{ij}^*(t, \eta_{ij}, \mathbf{x}_{[i+1,n]} + \epsilon \mathbf{e}_{i'}^{n-i}, \mathbf{y}_{[j+1,m]}) - a_{ij}^*(t, \eta_{ij}, \mathbf{x}_{[i+1,n]}, \mathbf{y}_{[j+1,m]}) \le a_{ij}^*(t, \eta_{ij}, \mathbf{x}_{[i+1,n]} + \epsilon \mathbf{e}_{i'+1}^{n-i}, \mathbf{y}_{[j+1,m]}) - a_{ij}^*(t, \eta_{ij}, \mathbf{x}_{[i+1,n]}, \mathbf{y}_{[j+1,m]}) \le 0$. By symmetry, we have the results with respect to the supply levels. □

Proof of Proposition 4. The result for $\beta = 0$ follows directly from Proposition 3. For $\alpha = \beta$, the result can be proved by applying the same approach as in Yu et al. (2015), thus we omit the lengthy details. □

Proof of Corollary 6. The dual problem of problem (s-p) is given by

$$\text{(s-d)}\quad \max_{p^d_{it},p^s_{jt}} \quad -\sum_{i=1}^{n} p^d_{it}x_{it} - \sum_{j=1}^{m} p^s_{jt}y_{jt}$$
$$\text{s.t.}\quad p^d_{it} + p^s_{jt} \geq \gamma^{t-1}(r_{ij}+c+h) + \alpha f^d_{it} + \beta f^s_{jt}, \quad 1\leq i\leq n, 1\leq j\leq m,$$
$$p^d_{it}, p^s_{it} \geq 0, \quad 1\leq i\leq n, 1\leq j\leq m.$$

The complementary slackness conditions are the same as in (CS) for period t. We see that the primal and dual optimal solutions to problems (P) and (D) for period t are primal- and dual-feasible for problems (s-p) and (s-d). Moreover, they satisfy the complementary slackness conditions of the subproblem. Thus, the optimal solution to (P) for period t is optimal for the subproblem (s-p). □

Proof of Proposition 6. Let Ω be the set of all sample paths of demand and supply realizations over the finite horizon, $\omega \in \Omega$ be a sample path, and $p(\omega)$ be the density at ω. We rewrite the stochastic model in the following form of a stochastic program.

$$\text{(P1)}\quad \max \quad \int_{\Omega} p(\omega)\sum_{t=1}^{T}\gamma^{t-1}\{\sum_{i=1}^{n}\sum_{j=1}^{m} r_{ij}q_{ijt}(\omega) - c\sum_{i=1}^{n}[x_{it}(\omega)-\sum_{j=1}^{m}q_{ijt}(\omega)] - h\sum_{j=1}^{m}[y_{jt}(\omega)-\sum_{i=1}^{n}q_{ijt}(\omega)]\}d\omega$$
$$\text{s.t.}\quad x_{i,t+1}(\omega) = \alpha[x_{it}(\omega) - \sum_{j=1}^{m}q_{ijt}(\omega)] + D_{it}(\omega), \text{ for all } 1\leq i\leq n, 1\leq t\leq T-1 \text{ and } \omega\in\Omega,$$
$$y_{j,t+1}(\omega) = \beta[y_{jt}(\omega) - \sum_{i=1}^{n}q_{ijt}(\omega)] + S_{jt}(\omega), \text{ for all } 1\leq j\leq m, 1\leq t\leq T-1 \text{ and } \omega\in\Omega,$$
$$\sum_{i=1}^{n}q_{ijt}(\omega)\leq y_{jt}(\omega) \text{ for all } 1\leq j\leq m, 1\leq t\leq T \text{ and } \omega\in\Omega,$$
$$\sum_{j=1}^{m}q_{ijt}(\omega)\leq x_{it}(\omega) \text{ for all } 1\leq i\leq n, 1\leq t\leq T \text{ and } \omega\in\Omega,$$
$$q_{ijt}(\omega)\geq 0 \text{ for all } 1\leq i\leq n, 1\leq j\leq m, 1\leq t\leq T \text{ and } \omega\in\Omega.$$

We denote by $Q^*_{ijt}(\omega)$, $\omega\in\Omega$, the optimal matching strategy, and by $x^*_{it}(\omega)$ and $y^*_{jt}(\omega)$ the associated state trajectory. Let $\bar{x}_{it}$, $\bar{y}_{jt}$ and $\bar{q}_{ijt}$ be the expectation of $x^*_{it}(\omega)$, $y^*_{jt}(\omega)$ and $q^*_{ijt}(\omega)$ over Ω, respectively. Because all sample paths satisfy the constraints of problem (P1), as expectations, $(\bar{q}_{ijt}, \bar{x}_{it}, \bar{y}_{jt})$ is *feasible* for the deterministic problem (P), with the corresponding objective value equal to the optimal value of the stochastic problem (P1). Therefore, the deterministic problem (P) has a larger optimal value than the stochastic problem (P1). □

Proof of Proposition ??? For $\varepsilon\geq 0$, we first consider the difference $V_t(\mathbf{x}+\varepsilon\mathbf{e}^n_i,\mathbf{y}) - V_t(\mathbf{x},\mathbf{y})$. An additional unit of demand contributes at most $\max_{i\in\mathcal{D},j\in\mathcal{S}} r_{ij} + \sum_{\tau=t}^{T}(\beta\gamma)^{\tau-t}h$ (reward from matching and possible saving in holding cost for reducing a unit of supply) to the total surplus, We have $V_{t+1}(\mathbf{x}+\varepsilon\mathbf{e}^n_i,\mathbf{y}) - V_{t+1}(\mathbf{x},\mathbf{y}) \leq (\max_{i\in\mathcal{D},j\in\mathcal{S}} r_{ij} + \sum_{\tau=t}^{T}(\beta\gamma)^{\tau-t}h)\varepsilon$. On the other hand, in the worst case, an additional unit of demand (or supply) incurs the extra waiting (or holding) cost $\sum_{\tau=t}^{T}(\alpha\gamma)^{\tau-t}c$ Thus, $V_{t+1}(\mathbf{x}+\varepsilon\mathbf{e}_i^n,\mathbf{y}) - V_{t+1}(\mathbf{x},\mathbf{y}) \geq -(\sum_{\tau=t}^{T}(\alpha\gamma)^{\tau-t})c\varepsilon$. Then, there exists $C>0$ such that $|V_{t+1}(\mathbf{x}+\varepsilon\mathbf{e}_i^n,\mathbf{y}) - V_{t+1}(\mathbf{x},\mathbf{y})| \leq C|\varepsilon|$. The same inequality holds holds for $\varepsilon < 0$ by replacing $\mathbf{x}$ with $\mathbf{x}-\mathbf{e}_i^n$. Likewise, we can show that $|V_{t+1}(\mathbf{x},\mathbf{y}+\varepsilon\mathbf{e}_j^n) - V_{t+1}(\mathbf{x},\mathbf{y})| \leq C|\varepsilon|$ for sufficiently large C. It follows that $|V_{t+1}(\mathbf{x}+\sum_{i=1}^n \varepsilon_i\mathbf{e}_i^n,\mathbf{y}+\sum_{j=1}^m \zeta_j\mathbf{e}_j^n) - V_{t+1}(\mathbf{x},\mathbf{y})| \leq \sum_{r=1}^n |V_t(\mathbf{x}+\sum_{i=1}^r \varepsilon_i\mathbf{e}_i^n,\mathbf{y}+\sum_{j=1}^m \zeta_j\mathbf{e}_j^n) - V_t(\mathbf{x}+\sum_{i=1}^{r-1} \varepsilon_i\mathbf{e}_i^n,\mathbf{y}+\sum_{j=1}^m \zeta_j\mathbf{e}_j^n)| + \sum_{s=1}^m |V_t(\mathbf{x},\mathbf{y}+\sum_{j=1}^s \zeta_j\mathbf{e}_j^n) - V_t(\mathbf{x},\mathbf{y}+\sum_{j=1}^{s-1} \zeta_j\mathbf{e}_j^n)| \leq C(\sum_{i=1}^n|\varepsilon_i| + \sum_{j=1}^m|\zeta_j|)$. Equivalently, we have $|V_t(\mathbf{x}',\mathbf{y}') - V_t(\mathbf{x},\mathbf{y})| \leq C||\mathbf{x}'-\mathbf{x}||_1 + C||\mathbf{y}'-\mathbf{y}||_1$, where $||z||_1 = \sum_{i=1}^r |z_i|$ for $\mathbf{z}=(z_1,\ldots,z_r)\in\mathbb{R}^r$.

The Proposition clearly holds for period $T+1$ since $V_{T+1}^k(\mathbf{x},\mathbf{y}) = V_{T+1}^{\det}(\mathbf{x},\mathbf{y}) \equiv 0$.

Suppose that $\lim_{k\to\infty} V_{t+1}^k(k\mathbf{x},k\mathbf{y})/k = V_{t+1}^{\det}(\mathbf{x},\mathbf{y})$ and the convergence is uniform with respect to $\mathbf{x}$ and $\mathbf{y}$. We want to show the same result holds for periopd t.

A matching decision $k\mathbf{Q}$ is feasible under the state $(k\mathbf{x},k\mathbf{y})$ if and only if $\mathbf{Q}$ is feasible under $(\mathbf{x},\mathbf{y})$. Let $\mathbf{u}=\mathbf{x}-\mathbf{1}_m\mathbf{Q}^\intercal$ and $\mathbf{v}=\mathbf{y}-\mathbf{1}_n\mathbf{Q}$ be the post-matching levels. Since

$$
\begin{aligned}
&|\frac{1}{k}V_{t+1}^k(\alpha k\mathbf{u}+\mathbf{D}_{t+1}^k,\beta k\mathbf{v}+\mathbf{S}_{t+1}^k) - V_{t+1}^{\det}(\alpha\mathbf{u}+\boldsymbol{\lambda}_{t+1},\beta\mathbf{v}+\boldsymbol{\mu}_{t+1})| \\
\leq &|\frac{1}{k}V_{t+1}^k(\alpha k\mathbf{u}+\mathbf{D}_{t+1}^k,\beta k\mathbf{v}+\mathbf{S}_{t+1}^k) - \frac{1}{k}V_{t+1}^k(\alpha k\mathbf{u}+k\boldsymbol{\lambda}_{t+1},\beta k\mathbf{v}+k\boldsymbol{\mu}_{t+1})| \\
&+|\frac{1}{k}V_{t+1}^k(\alpha k\mathbf{u}+k\boldsymbol{\lambda}_{t+1},\beta k\mathbf{v}+k\boldsymbol{\mu}_{t+1}) - V_{t+1}^{\det}(\alpha\mathbf{u}+\boldsymbol{\lambda}_{t+1},\beta\mathbf{v}+\boldsymbol{\mu}_{t+1})| \\
\leq &C||\frac{\mathbf{D}_{t+1}^k}{k}-\boldsymbol{\lambda}_{t+1}||_1 + |\frac{1}{k}V_{t+1}^k(\alpha k\mathbf{u}+k\boldsymbol{\lambda}_{t+1},\beta k\mathbf{v}+\boldsymbol{\mu}_{t+1}^k) - V_{t+1}^{\det}(\alpha\mathbf{u}+\boldsymbol{\lambda}_{t+1},\beta\mathbf{v}+\boldsymbol{\mu}_{t+1})|. \qquad (12)
\end{aligned}
$$

The first term on the right-hand side converges to 0 with probability 1 as $k\to\infty$ by the strong law of large numbers, and the second term also converges to 0 by induction. Therefore, $[V_{t+1}^k(\alpha k\mathbf{u}+\mathbf{D}_{t+1}^k,\beta k\mathbf{v}+\mathbf{S}_{t+1}^k)]/k \to V_{t+1}^{\det}(\alpha\mathbf{u}+\boldsymbol{\lambda}_{t+1},\beta\mathbf{v}+\boldsymbol{\mu}_{t+1})$ with probability 1. Moreover,

$$
\begin{aligned}
&E||\mathbf{D}_{t+1}^k/k-\boldsymbol{\lambda}_{t+1}||_1 \\
=&E\left[||\mathbf{D}_{t+1}^k/k-\boldsymbol{\lambda}_{t+1}||_1 1\left\{||\mathbf{D}_{t+1}^k/k-\boldsymbol{\lambda}_{t+1}||_1 > B\right\}\right] + E\left[||\mathbf{D}_{t+1}^k/k-\boldsymbol{\lambda}_{t+1}||_1 1\left\{||\mathbf{D}_{t+1}^k/k-\boldsymbol{\lambda}_{t+1}||_1 \leq B\right\}\right] \\
\leq&\frac{1}{k}\sum_{\ell=1}^k E\left[||\mathbf{D}_{t+1}(\ell)-\boldsymbol{\lambda}_{t+1}||_1 1\left\{||\mathbf{D}_{t+1}^k/k-\boldsymbol{\lambda}_{t+1}||_1 > B\right\}\right] + E\left[||\mathbf{D}_{t+1}^k/k-\boldsymbol{\lambda}_{t+1}||_1 1\left\{||\mathbf{D}_{t+1}^k/k-\boldsymbol{\lambda}_{t+1}||_1 \leq B\right\}\right] \\
=&E\left[||\mathbf{D}_{t+1}(1)-\boldsymbol{\lambda}_{t+1}||_1 1\left\{||\mathbf{D}_{t+1}^k/k-\boldsymbol{\lambda}_{t+1}||_1 > B\right\}\right] + E\left[||\mathbf{D}_{t+1}^k/k-\boldsymbol{\lambda}_{t+1}||_1 1\left\{||\mathbf{D}_{t+1}^k/k-\boldsymbol{\lambda}_{t+1}||_1 \leq B\right\}\right],
\end{aligned}
$$

where the second equality follows from symmetry. For $B>0$, $1\{||\mathbf{D}_{t+1}^k/k-\boldsymbol{\lambda}_{t+1}||_1 > B\} \to 0$ as $k\to\infty$ because $\mathbf{D}_{t+1}^k/k-\boldsymbol{\lambda}_{t+1}\to 0$ by the strong law of large numbers. Given that $\mathbf{D}_{t+1}(1)$ is integrable, and $||\mathbf{D}_{t+1}^k/k-\boldsymbol{\lambda}_{t+1}||_1 1\{||\mathbf{D}_{t+1}^k/k-\boldsymbol{\lambda}_{t+1}||_1 \leq B\} \leq B$, we can apply the dominant convergence theorem and see that both terms on the right-hand side converges to 0. Thus, $E||\mathbf{D}_{t+1}^k/k-\boldsymbol{\lambda}_{t+1}||_1 \to 0$

as $k\to\infty$. By (12), $EV_{t+1}^k(\alpha k\mathbf{u}+\mathbf{D}_{t+1}^k,\beta k\mathbf{v}+\mathbf{S}_{t+1}^k)/k\to V_{t+1}^{\det}(\alpha\mathbf{u}+\boldsymbol{\lambda}_{t+1},\beta\mathbf{v}+\boldsymbol{\mu}_{t+1})$, and the convergence is uniform as long as the convergence $|\frac{1}{k}V_{t+1}^k(\alpha k\mathbf{u}+k\boldsymbol{\lambda}_{t+1},\beta k\mathbf{v}+\boldsymbol{\mu}_{t+1}^k)-V_{t+1}^{\det}(\alpha\mathbf{u}+\boldsymbol{\lambda}_{t+1},\beta\mathbf{v}+\boldsymbol{\mu}_{t+1})|\to 0$ is uniform, which is ensured by the induction hypothesis.

It then follows that $H_t^k(k\mathbf{Q},k\mathbf{x},k\mathbf{y})/k=\mathbf{r}\circ\mathbf{Q}+c\mathbf{1}_n(\mathbf{x}^\intercal-\mathbf{Q})\mathbf{1}_m^\intercal+h\mathbf{1}_m(\mathbf{y}-\mathbf{1}_n\mathbf{Q})+\gamma EV_{t+1}^k(\alpha k\mathbf{u}+\mathbf{D}_{t+1}^k,\beta k\mathbf{v}+\mathbf{S}_{t+1}^k)/k\to\mathbf{r}\circ\mathbf{Q}+c\mathbf{1}_n(\mathbf{x}^\intercal-\mathbf{Q})\mathbf{1}_m^\intercal+h\mathbf{1}_m(\mathbf{y}-\mathbf{1}_n\mathbf{Q})+\gamma V_{t+1}^{\det}(\alpha\mathbf{u}+\boldsymbol{\lambda}_{t+1},\beta\mathbf{v}+\boldsymbol{\mu}_{t+1})=H_t^{\det}(\mathbf{Q},\mathbf{x},\mathbf{y})$ uniformly with respect to $\mathbf{Q}$, $\mathbf{x}$ and $\mathbf{y}$. Thus, for $\epsilon>0$, there exists K (independent of $\mathbf{Q}$, $\mathbf{x}$ and $\mathbf{y}$) such that $|H_t^k(k\mathbf{Q},k\mathbf{x},k\mathbf{y})/k-H_t^{\det}(\mathbf{Q},\mathbf{x},\mathbf{y})|<\epsilon$ for $k>K$. Let $\mathbf{Q}^{\det *}$ be the optimal decision in period t for the deterministic system. We have $V_t^k(k\mathbf{x},k\mathbf{y})/k\ge H_t^k(k\mathbf{Q}^{\det *},k\mathbf{x},k\mathbf{y})/k\ge H_t^{\det}(\mathbf{Q}^{\det *},\mathbf{x},\mathbf{y})-\epsilon=V_t^{\det}(\mathbf{x},\mathbf{y})-\epsilon$. Moreover, for any $\mathbf{Q}$ feasible under $(\mathbf{x},\mathbf{y})$, $V_t^{\det}(\mathbf{x},\mathbf{y})\ge H_t^{\det}(\mathbf{Q},\mathbf{x},\mathbf{y})\ge H_t(k\mathbf{Q},k\mathbf{x},k\mathbf{y})/k-\epsilon$ for $k>K$. Thus, $V_t^{\det}(\mathbf{x},\mathbf{y})\ge\max_{\mathbf{Q}}H_t^k(k\mathbf{Q},k\mathbf{x},k\mathbf{y})/k-\epsilon=V_t^k(k\mathbf{x},k\mathbf{y})/k-\epsilon$. Therefore, $|V_t^{\det}(\mathbf{x},\mathbf{y})-V_t^k(k\mathbf{x},k\mathbf{y})/k|<\epsilon$ for $k>K$ and K is independent of $\mathbf{Q}$, $\mathbf{x}$ and $\mathbf{y}$. This shows that $\lim_{k\to\infty}V_t^k(k\mathbf{x},k\mathbf{y})/k=V_t^{\det}(\mathbf{x},\mathbf{y})$ uniformly. □

Proof of Proposition 7. We first prove the following lemma.

LEMMA B.1. *Suppose that demand and supply of all types are bounded; i.e., there exist* $\mathbf{U}^d\in\mathbb{R}^n$ *and* $\mathbf{U}^s\in\mathbb{R}^m$ *such that* $\mathbf{D}_t\le\mathbf{U}^d$ *and* $\mathbf{S}_t\le\mathbf{U}^s$ *almost surely for* $t=1,\dots,T$. *Then,* $\lim_{k\to\infty}V_t^k(\mathbf{x},\mathbf{y})=V_t^{\det}(\mathbf{x},\mathbf{y})$ *for* $t=1,2,\dots,T+1$. *Furthermore, for any compact set* $B\in\mathbb{R}^{n+m}$, *the convergence is uniform over* $(\mathbf{x},\mathbf{y})\in B$.

Proof of Lemma B.1. The functions $H_t^k(\mathbf{Q},\mathbf{x},\mathbf{y})$ are defined in the same way as we defined $H_t(\mathbf{Q},\mathbf{x},\mathbf{y})$.

We prove this proposition by induction on t. The result is clearly true for $t=T+1$ because $V_{T+1}^k(\mathbf{x},\mathbf{y})\equiv V_{T+1}^{\det}(\mathbf{x},\mathbf{y})\equiv 0$. Suppose that for any compact set $B\in\mathbb{R}^{n+m}$, $V_{t+1}^k(\mathbf{x},\mathbf{y})\to V_{t+1}^{\det}(\mathbf{x},\mathbf{y})$ uniformly for $(\mathbf{x},\mathbf{y})\in B$ as $k\to\infty$. Now consider an arbitary compact set $B_0\in\mathbb{R}^{n+m}$. Since demand and supply are bounded, there exists a compact set B_1 such that $(\alpha\mathbf{u}+\bar{\mathbf{D}}_{t+1}^k,\beta\mathbf{v}+\bar{\mathbf{S}}_{t+1}^k)\in B_1$ and $(\alpha\mathbf{u}+\boldsymbol{\lambda}_{t+1},\beta\mathbf{v}+\boldsymbol{\mu}_{t+1})\in B_1$ if $(\mathbf{x},\mathbf{y})\in B_0$, where $\mathbf{u}=\mathbf{x}-\mathbf{1}^m\mathbf{Q}^\intercal$ and $\mathbf{v}=\mathbf{y}-\mathbf{1}^n\mathbf{Q}$. By the induction hypothesis, for $\forall\epsilon>0$, there exists K such that $|V_{t+1}^k(\alpha\mathbf{u}+\bar{\mathbf{D}}_{t+1}^k,\beta\mathbf{v}+\bar{\mathbf{S}}_{t+1}^k)-V_{t+1}^{\det}(\alpha\mathbf{u}+\bar{\mathbf{D}}_{t+1}^k,\beta\mathbf{v}+\bar{\mathbf{S}}_{t+1}^k)|<\frac{\epsilon}{2}$ for $k\ge K$ over all $(\mathbf{x},\mathbf{y})\in B_0$ and all realizations of demand and supply.

On the other hand, since $V_{t+1}^{\det}(\mathbf{x},\mathbf{y})$ is continuous, it is bounded and uniformly continuous over $(\mathbf{x},\mathbf{y})\in B_1$. Suppose that $|V_{t+1}^{\det}(\mathbf{x},\mathbf{y})|\le M$ for $(\mathbf{x},\mathbf{y})\in B_1$. Uniform continuity implies the existence of $\delta>0$ such that $|V_{t+1}^{\det}(\mathbf{x},\mathbf{y})-V_{t+1}^{\det}(\mathbf{x}',\mathbf{y}')|<\frac{\epsilon}{4}$ as long as $\{(\mathbf{x},\mathbf{y}),(\mathbf{x}',\mathbf{y}')\}\subset B_1$ and $\|(\mathbf{x},\mathbf{y})-(\mathbf{x}',\mathbf{y}')\|<\delta$. Here, $\|\cdot\|$ can be any norm on $\mathbb{R}^{n+m}$. We have

$$
\begin{aligned}
&|EV_{t+1}^{\det}(\alpha\mathbf{u}+\bar{\mathbf{D}}_{t+1}^k,\beta\mathbf{v}+\bar{\mathbf{S}}_{t+1}^k)-EV_{t+1}^{\det}(\alpha\mathbf{u}+\lambda_{t+1},\beta\mathbf{v}+\mu_{t+1})|\\
\le&\left|E\left\{\left[V_{t+1}^{\det}(\alpha\mathbf{u}+\bar{\mathbf{D}}_{t+1}^k,\beta\mathbf{v}+\bar{\mathbf{S}}_{t+1}^k)-V_{t+1}^{\det}(\alpha\mathbf{u}+\boldsymbol{\lambda}_{t+1},\beta\mathbf{v}+\boldsymbol{\mu}_{t+1})\right]1_{\{\|(\bar{\mathbf{D}}_{t+1}^k,\bar{\mathbf{S}}_{t+1}^k)-(\boldsymbol{\lambda}_{t+1},\boldsymbol{\mu}_{t+1})\|<\delta\}}\right\}\right|
\end{aligned}
$$

$$+\left|E\left\{\left[V_{t+1}^{\det}(\alpha\mathbf{u}+\bar{\mathbf{D}}_{t+1}^{k},\beta\mathbf{v}+\bar{\mathbf{S}}_{t+1}^{k})-V_{t+1}^{\det}(\alpha\mathbf{u}+\lambda_{t+1},\beta\mathbf{v}+\mu_{t+1})\right]1_{\left\{\|(\bar{\mathbf{D}}_{t+1}^{k},\bar{\mathbf{S}}_{t+1}^{k})-(\boldsymbol{\lambda}_{t+1},\boldsymbol{\mu}_{t+1})\|\geq\delta\right\}}\right\}\right|$$
$$\leq\frac{\epsilon}{4}+2M\cdot E\left\{1_{\left\{\|(\bar{\mathbf{D}}_{t+1}^{k},\bar{\mathbf{S}}_{t+1}^{k})-(\boldsymbol{\lambda},\boldsymbol{\mu})\|\geq\delta\right\}}\right\}=\frac{\epsilon}{4}+2M\cdot\Pr(\|(\bar{\mathbf{D}}^{k},\bar{\mathbf{S}}_{t+1}^{k})-(\boldsymbol{\lambda}_{t+1},\boldsymbol{\mu}_{t+1})\|\geq\delta). \tag{13}$$

By the weak law of large numbers, $\Pr(\|(\bar{\mathbf{D}}_{t+1}^{k},\bar{\mathbf{S}}_{t+1}^{k})-(\boldsymbol{\lambda}_{t+1},\boldsymbol{\mu}_{t+1})\|\geq\delta)\to 0$ as $k\to\infty$. Then, we can find a sufficiently large $K_1\geq K$ such that $\Pr(\|(\bar{\mathbf{D}}_{t+1}^{k},\bar{\mathbf{S}}_{t+1}^{k})-(\boldsymbol{\lambda}_{t+1},\boldsymbol{\mu}_{t+1})\|\geq\delta)<\epsilon/(8M)$ when $k\geq K_1$. Combining this with (13), we have $|EV_{t+1}^{\det}(\alpha\mathbf{u}+\bar{\mathbf{D}}_{t+1}^{k},\beta\mathbf{v}+\bar{\mathbf{S}}_{t+1}^{k})-EV_{t+1}^{\det}(\alpha\mathbf{u}+\boldsymbol{\lambda}_{t+1},\beta\mathbf{v}+\boldsymbol{\mu}_{t+1})|<\frac{\epsilon}{2}$ when $k\geq K_1$. Thus, $|EV_{t+1}^{k}(\alpha\mathbf{u}+\bar{\mathbf{D}}_{t+1}^{k},\beta\mathbf{v}+\bar{\mathbf{S}}_{t+1}^{k})-EV_{t+1}^{\det}(\alpha\mathbf{u}+\boldsymbol{\lambda}_{t+1},\beta\mathbf{v}+\boldsymbol{\mu}_{t+1})|\leq E|V_{t+1}^{k}(\alpha\mathbf{u}+\bar{\mathbf{D}}_{t+1}^{k},\beta\mathbf{v}+\bar{\mathbf{S}}_{t+1}^{k})-V_{t+1}^{\det}(\alpha\mathbf{u}+\bar{\mathbf{D}}_{t+1}^{k},\beta\mathbf{v}+\bar{\mathbf{S}}_{t+1}^{k})|+|EV_{t+1}^{\det}(\alpha\mathbf{u}+\bar{\mathbf{D}}_{t+1}^{k},\beta\mathbf{v}+\bar{\mathbf{S}}_{t+1}^{k})-EV_{t+1}^{\det}(\alpha\mathbf{u}+\boldsymbol{\lambda}_{t+1},\beta\mathbf{v}+\boldsymbol{\mu}_{t+1})|<\epsilon$ for $k>K_1$, regardless of the choice of $(\mathbf{x},\mathbf{y})$ in B_0. This immediately implies that $|H_t^k(\mathbf{Q},\mathbf{x},\mathbf{y})-H_t^{\det}(\mathbf{Q},\mathbf{x},\mathbf{y})|<\epsilon$ if $k>K_1$, over all $(\mathbf{x},\mathbf{y})\in B_0$ and feasible $\mathbf{Q}$. Then, $V_t^k(\mathbf{x},\mathbf{y})=\max_{\mathbf{Q}\text{ is feasible}}H_t^k(\mathbf{Q},\mathbf{x},\mathbf{y})<\max_{\mathbf{Q}\text{ is feasible}}\{H_t^{\det}(\mathbf{Q},\mathbf{x},\mathbf{y})+\epsilon\}=\max_{\mathbf{Q}\text{ is feasible}}H_t^{\det}(\mathbf{Q},\mathbf{x},\mathbf{y})+\epsilon=V^{\det}(\mathbf{x},\mathbf{y})+\epsilon$ for all $(\mathbf{x},\mathbf{y})\in B_0$ if $k\geq K_1$. By exchanging the roles of V_t^k and $V_t^{\det}$, we can show that $V_t^{\det}(\mathbf{x},\mathbf{y})\leq V_t^k(\mathbf{x},\mathbf{y})+\epsilon$ for all $(\mathbf{x},\mathbf{y})\in B_0$ if $k\geq K_1$. Therefore, $V_t^k(\mathbf{x},\mathbf{y})\to V_t^{\det}(\mathbf{x},\mathbf{y})$ uniformly as $k\to\infty$ for all $(\mathbf{x},\mathbf{y})\in B_0$. $\square$

The proof is again by induction on t. It is clear that it holds for $t=T+1$. Suppose it holds for $t+1$.

As in the proof of Lemma B.1, we first bound the difference $|EV_{t+1}^{\det}(\alpha\mathbf{u}+\bar{\mathbf{D}}_{t+1}^{k},\beta\mathbf{v}+\bar{\mathbf{S}}_{t+1}^{k})-EV_{t+1}^{\det}(\alpha\mathbf{u}+\boldsymbol{\lambda}_{t+1},\beta\mathbf{v}+\boldsymbol{\mu}_{t+1})|<\frac{\epsilon}{2}$.

Consider positive vectors $\mathbf{U}^d=(U^d,\ldots,U^d)\in\mathbb{R}^n$ and $\mathbf{U}^s=(U^s,\ldots,U^s)\in\mathbb{R}^m$. We define $\bar{\mathbf{D}}_{t+1}^{k}(U^d)\overset{\text{def}}{=}\bar{\mathbf{D}}_{t+1}^{k}\wedge\mathbf{U}^d=(\bar{D}_{1,t+1}^{k}\wedge U^d,\ldots,\bar{D}_{n,t+1}^{k}\wedge U^d)$ and $\bar{\mathbf{S}}_{t+1}^{k}(U^s)\overset{\text{def}}{=}\bar{\mathbf{S}}_{t+1}^{k}\wedge\mathbf{U}^s=(\bar{S}_{1,t+1}^{k}\wedge U^s,\ldots,\bar{S}_{m,t+1}^{k}\wedge U^s)$ as the truncated demand and supply. Clearly, the truncated demand and supply is weakly less than the origial demand and supply. Further, we let $\boldsymbol{\lambda}_{t+1}(U^d)=\boldsymbol{\lambda}_{t+1}\wedge\mathbf{U}^d$ and $\boldsymbol{\mu}_{t+1}(U^s)=\boldsymbol{\mu}_{t+1}\wedge\mathbf{U}^s$. By the strong law of large numbers, $\bar{\mathbf{D}}_{t+1}^{k}(U^d)=\bar{\mathbf{D}}_{t+1}^{k}\wedge\mathbf{U}^d\to\boldsymbol{\lambda}_{t+1}\wedge\mathbf{U}^d=\boldsymbol{\lambda}(U^d)$ almost surely and $\bar{\mathbf{S}}_{t+1}^{k}(U^d)\to\boldsymbol{\mu}_{t+1}(U^s)$ almost surely as $k\to\infty$. It is also clear that $\boldsymbol{\lambda}_{t+1}(U^d)=\boldsymbol{\lambda}_{t+1}$ when U^d is sufficiently large (larger than $\max_{1\leq i\leq n}\lambda_{i,t+1}$) and $\boldsymbol{\mu}_{t+1}(U^s)=\boldsymbol{\mu}_{t+1}$ when U^s is sufficiently large. Let B_0 and B_1 be compact sets defined as in Lemma B.1.

Consider the difference $V_{t+1}^k(\mathbf{x},\mathbf{y})-V_{t+1}^k(\mathbf{x}',\mathbf{y}')$ with $(\mathbf{x},\mathbf{y})\geq(\mathbf{x}',\mathbf{y}')$. Note that an additional unit of demand contributes at most $\max_{i\in\mathcal{D},j\in\mathcal{S}}r_{ij}+\sum_{\tau=t}^{T}(\beta\gamma)^{\tau-t}h$ (reward from matching and possible saving in holding cost for reducing a unit of supply) to the total surplus, and an addition unit of supply contributes at most $\max_{i\in\mathcal{D},j\in\mathcal{S}}r_{ij}+\sum_{\tau=t}^{T}(\alpha\gamma)^{\tau-t}c$ to the total surplus. We have $V_{t+1}(\mathbf{x},\mathbf{y})-V_{t+1}(\mathbf{x}',\mathbf{y}')\leq(\max_{i\in\mathcal{D},j\in\mathcal{S}}r_{ij}+\sum_{\tau=t}^{T}(\beta\gamma)^{\tau-t}h)\sum_{i=1}^{n}(x_i-x_i')+(\max_{i\in\mathcal{D},j\in\mathcal{S}}r_{ij}+\sum_{\tau=t}^{T}(\alpha\gamma)^{\tau-t}c)\sum_{j=1}^{m}(y_j-y_j')$. On the other hand, in the worst case, an additional unit of demand (or supply) incurs the extra waiting (or holding) cost $\sum_{\tau=t}^{T}(\alpha\gamma)^{\tau-t}c$ (or $\sum_{\tau=t}^{T}(\beta\gamma)^{\tau-t}h$). This implies that $V^k_{t+1}(\mathbf{x},\mathbf{y}) - V^k_{t+1}(\mathbf{x}',\mathbf{y}') \geq -(\sum_{\tau=t}^T (\alpha\gamma)^{\tau-t} c)\sum_{i=1}^n (x_i - x_i') - (\sum_{\tau=t}^T (\beta\gamma)^{\tau-t} h)\sum_{j=1}^m (y_j - y_j')$. From the above arguments, there exists constant $C > 0$ such that $|V^k_{t+1}(\mathbf{x},\mathbf{y}) - V^k_{t+1}(\mathbf{x}',\mathbf{y}')| \leq C[\sum_{i=1}^n (x_i - x_i') + \sum_{j=1}^m (y_j - y_j')]$. Then,

$$\begin{aligned}
&E|V^k_{t+1}(\alpha\mathbf{u} + \bar{\mathbf{D}}^k_{t+1}, \beta\mathbf{v} + \bar{\mathbf{S}}^k_{t+1}) - V^k_{t+1}(\alpha\mathbf{u} + \bar{\mathbf{D}}^k_{t+1}(U^d), \beta\mathbf{v} + \bar{\mathbf{S}}^k_{t+1}(U^s))| \\
\leq & C \cdot E\left\{\sum_{i=1}^n [\bar{D}^k_{i,t+1} - \bar{D}^k_{i,t+1}(U^d)] + \sum_{j=1}^m [\bar{S}^k_{j,t+1} - \bar{S}^k_{j,t+1}(U^s)]\right\} \\
= & C\sum_{i=1}^n E[\bar{D}^k_{i,t+1} - \bar{D}^k_{i,t+1}(U^d)] + C\sum_{j=1}^m E[\bar{S}^k_{j,t+1} - \bar{S}^k_{j,t+1}(U^s)] \\
= & C\sum_{i=1}^n E[(\bar{D}^k_{i,t+1} - U^d)1_{\{\bar{D}^k_{i,t+1} - U^d > 0\}}] + C\sum_{j=1}^m E[(\bar{S}^k_{j,t+1} - U^s)1_{\{\bar{S}^k_{j,t+1} - U^s > 0\}}].
\end{aligned}$$

For any convex function f, $Ef(\bar{D}^k_{i,t+1} - U^d) = Ef(\frac{1}{k}\sum_{\ell=1}^k (D^\ell_{i,t+1} - U^d)) \leq \frac{1}{k}\sum_{\ell=1}^k Ef(D^\ell_{i,t+1} - U^d)) = Ef(D_{i,t+1} - U^d)$. Thus, $\bar{D}^k_{i,t+1} - U^d$ is dominated by $D_{i,t+1} - U^d$ in convex order. Since the function $x1_{\{x>0\}}$ is convex, we have $E[(\bar{D}^k_{i,t+1} - U^d)1_{\{\bar{D}^k_{i,t+1} - U^d > 0\}}] \leq E[(D_{i,t+1} - U^d)1_{\{D_{i,t+1} - U^d > 0\}}]$. Likewise, $E[(\bar{S}^k_{j,t+1} - U^s)1_{\{\bar{S}^k_{j,t+1} - U^s > 0\}}] \leq E[(S_{j,t+1} - U^s)1_{\{S_{j,t+1} - U^s > 0\}}]$. By the dominated convergence theorem, we know that $\lim_{U^d \to \infty} E[(D_{i,t+1} - U^d)1_{\{D_{i,t+1} - U^d > 0\}}] = E[\lim_{U^d \to \infty}(D_{i,t+1} - U^d)1_{\{D_{i,t+1} - U^d > 0\}}] = 0$ and analogously, $\lim_{U^s \to \infty} E[(S_{j,t+1} - U^s)1_{\{S_{j,t+1} - U^s > 0\}}] = 0$. This implies that $E|V^k_{t+1}(\alpha\mathbf{u} + \bar{\mathbf{D}}^k_{t+1}, \beta\mathbf{v} + \bar{\mathbf{S}}^k_{t+1}) - V^k_{t+1}(\alpha\mathbf{u} + \bar{\mathbf{D}}^k_{t+1}(U^d), \beta\mathbf{v} + \bar{\mathbf{S}}^k_{t+1}(U^s))|$ converges to zero uniformly as $k \to \infty$.

Consequently, for $\forall \epsilon > 0$, there exists sufficiently large U (independent of k) such that when $U^d > U$ and $U^s > U$, $E|V^k_{t+1}(\alpha\mathbf{u} + \bar{\mathbf{D}}^k_{t+1}, \beta\mathbf{v} + \bar{\mathbf{S}}^k_{t+1}) - V^k_{t+1}(\alpha\mathbf{u} + \bar{\mathbf{D}}^k_{t+1}(U^d), \beta\mathbf{v} + \bar{\mathbf{S}}^k_{t+1}(U^s))| < \frac{\epsilon}{2}$.

We choose $U^d > U \wedge (\max_{1\leq i\leq n} \lambda_{i,t+1})$ and $U^s > U \wedge (\max_{1\leq j\leq m} \mu_{j,t+1})$. Then, $\boldsymbol{\lambda}_{t+1}(U^d) = \boldsymbol{\lambda}_{t+1}$ and $\boldsymbol{\mu}_{t+1}(U^s) = \boldsymbol{\mu}_{t+1}$. Applying the analysis in Lemma B.1 for bounded demand (noting that now $\bar{\mathbf{D}}^k_{t+1}(U^d)$ and $\bar{\mathbf{S}}^k_{t+1}(U^s)$ are bounded), for the chosen U^d and U^s, there exists K (dependent on the choice of U^d and U^s) such that $|V^k_{t+1}(\alpha\mathbf{u} + \bar{\mathbf{D}}^k_{t+1}(U^d), \beta\mathbf{v} + \bar{\mathbf{S}}^k_{t+1}(U^s)) - V^{\det}_{t+1}(\alpha\mathbf{u} + \boldsymbol{\lambda}_{t+1}(U^d), \beta\mathbf{v} + \boldsymbol{\mu}_{t+1}(U^s))| < \frac{\epsilon}{2}$ when $k \geq K$. Then,

$$\begin{aligned}
&|EV^k_{t+1}(\alpha\mathbf{u} + \bar{\mathbf{D}}^k_{t+1}, \beta\mathbf{v} + \bar{\mathbf{S}}^k_{t+1}) - EV^{\det}_{t+1}(\alpha\mathbf{u} + \boldsymbol{\lambda}_{t+1}, \beta\mathbf{v} + \boldsymbol{\mu}_{t+1})| \\
= & |EV^k_{t+1}(\alpha\mathbf{u} + \bar{\mathbf{D}}^k_{t+1}, \beta\mathbf{v} + \bar{\mathbf{S}}^k_{t+1}) - EV^{\det}_{t+1}(\alpha\mathbf{u} + \boldsymbol{\lambda}_{t+1}(U^d), \beta\mathbf{v} + \boldsymbol{\mu}_{t+1}(U^s))| \\
\leq & E|V^k_{t+1}(\alpha\mathbf{u} + \bar{\mathbf{D}}^k_{t+1}, \beta\mathbf{v} + \bar{\mathbf{S}}^k_{t+1}) - V^k_{t+1}(\alpha\mathbf{u} + \bar{\mathbf{D}}^k_{t+1}(U^d), \beta\mathbf{v} + \bar{\mathbf{S}}^k_{t+1}(U^s))| \\
& + E|V^k_{t+1}(\alpha\mathbf{u} + \bar{\mathbf{D}}^k_{t+1}(U^d), \beta\mathbf{v} + \bar{\mathbf{S}}^k_{t+1}(U^s)) - V^{\det}_{t+1}(\alpha\mathbf{u} + \boldsymbol{\lambda}_{t+1}(U^d), \beta\mathbf{v} + \boldsymbol{\mu}_{t+1}(U^s))| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Then it follows that $|H^k_t(\mathbf{Q},\mathbf{x},\mathbf{y}) - H^{\det}_t(\mathbf{Q},\mathbf{x},\mathbf{y})| < \epsilon$ when $k \geq K$, for any $(\mathbf{x},\mathbf{y}) \in B_0$ and feasible $\mathbf{Q}$. The rest of the proof is the same as in Lemma B.1. □

Proof of Theorem 8. From the proof of Lemma B.1 and Proposition 7, for any $\epsilon > 0$, there exists $K > 0$ such that $|H_t^k(k\mathbf{Q}, k\mathbf{x}, k\mathbf{y})/k - H_t^{\det}(\mathbf{Q}, \mathbf{x}, \mathbf{y})| < \frac{\epsilon}{2}$ if $k \geq K$, and such K is independent of $\mathbf{Q}$, $\mathbf{x}$ and $\mathbf{y}$. Then, for any $\mathbf{Q}$ that is feasible under $\mathbf{x}, \mathbf{y})$, $H_t^k(k\hat{\mathbf{Q}}, k\mathbf{x}, k\mathbf{y})/k \geq H_t^{\det}(\hat{\mathbf{Q}}, \mathbf{x}, \mathbf{y}) - \frac{\epsilon}{2} \geq H_t^{\det}(\mathbf{Q}, \mathbf{x}, \mathbf{y}) - \frac{\epsilon}{2} \geq H_t^k(k\mathbf{Q}, k\mathbf{x}, k\mathbf{y})/k - \epsilon$ if $k \geq K$, where the second inequality follows from the optimality of $\hat{\mathbf{Q}}$. Thus, $H_t^k(k\hat{\mathbf{Q}}, k\mathbf{x}, k\mathbf{y})/k \geq \max_{\mathbf{Q}} H_t^k(k\mathbf{Q}, k\mathbf{x}, k\mathbf{y})/k - \epsilon = V_t^k(k\mathbf{x}, k\mathbf{y})/k$. □

C. One-Step-Ahead Heuristic

We show that the one-step-ahead heuristic policy for period t is determined by a state-dependent target level $\hat{b}_{ij}(t, \tilde{y}_m - \tilde{x}_n)$ and two other quantities $\overline{x}_i + \sum_{i'=i+1}^n x_{i'}$ and $\underline{x}_i + \sum_{i'=i+1}^n x_{i'}$. Recall that $\overline{x}_i = x_i - (\tilde{y}_{j-1} - \tilde{x}_{i-1})^+$ is the available quantity of type i demand before we consider the matching of type i demand with type j supply, and $\underline{x}_i = (\tilde{x}_i - \tilde{y}_j)^+$ is the post-matching level of type i demand immediately after we match type i demand with type j supply as much as possible. Then, $\overline{x}_i + \sum_{i'=i+1}^n x_{i'}$ is the available quantity in all the demand types before we match type i demand with type j supply, and $\underline{x}_i + \sum_{i'=i+1}^n x_{i'}$ is the aggregate post-matching level of all the demand types immediately after we match type i demand with type j demand as much as possible. We show the optimal matching policy has the following structure: If $\hat{b}_{ij}(t, \tilde{y}_m - \tilde{x}_n) \geq \overline{x}_i + \sum_{i'=i+1}^n x_{i'}$, it is optimal not to match type i demand with type j supply. If $\underline{x}_i + \sum_{i'=i+1}^n x_{i'} < \hat{b}_{ij}(t, \tilde{y}_m - \tilde{x}_n) < \overline{x}_i + \sum_{i'=i+1}^n x_{i'}$, it is optimal to reduce type i demand to the target level $\hat{b}_{ij}(t, \tilde{y}_m - \tilde{x}_n)$. If $\hat{b}_{ij}(t, \tilde{y}_m - \tilde{x}_n) < \underline{x}_i + \sum_{i'=i+1}^n x_{i'}$, it is optimal to match type i demand with type j supply as much as possible.

Theorem C.1 (One-step-ahead heuristic). *Suppose $\alpha = \beta$ and consider the one-step-ahead heuristic, which also follows the top-down matching procedure. In this procedure, consider matching type i demand with type j supply, which is optimal only if $\tilde{x}_i > \tilde{y}_{j-1}$ and $\tilde{x}_{i-1} < \tilde{y}_j$. There exists a target level $\hat{b}_{ij}(t, \tilde{y}_m - \tilde{x}_n)$ such that the optimal protection level for matching type i demand with type j supply is $a_{ij}^*(t, \mathbf{x}, \mathbf{y}) = [\hat{b}_{ij}(t, \tilde{y}_m - \tilde{x}_n) - (\underline{x}_i + \sum_{i'=i+1}^n x_{i'})]^+$. Moreover, the target level $\hat{b}_{ij}(t, \tilde{y}_m - \tilde{x}_n)$ for the aggregate post-matching demand is weakly decreasing in $\tilde{y}_m - \tilde{x}_n$ and the decreasing rate is dominated by 1.*

Proof of Theorem C.1. As part of the one-step-ahead heuristic, the greedy matching policy is implemented from period $t+1$. We first prove two lemmas on the greedy matching policy. With vertically differentiated types, the greedy matching policy also follows the top-down structure but does not reserve demand or supply. Let $V_t^g(\mathbf{x}, \mathbf{y})$ be the expected total discounted surplus under the greedy matching policy from the current period t to the end of the horizon, given the current state $(\mathbf{x}, \mathbf{y})$. The transformed state $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ is defined as in the previous subsection.

LEMMA C.1. *For $\epsilon > 0$ and $\delta_k \geq 0$, $k = 1, \ldots, j$ such that $\sum_{k=1}^{j} \delta_k = \epsilon$, the difference $V_t^g(\mathbf{x}, \mathbf{y} - \sum_{k=1}^{j} \delta_k \mathbf{e}_k + \epsilon \mathbf{e}_j) - V_t^g(\mathbf{x}, \mathbf{y})$ depends only on δ_k, $k = 1, \ldots, j$, $\tilde{x}_n$ and $\mathbf{y}_{[1,j-1]}$. Symmetrically, $V_t^g(\mathbf{x} - \sum_{k=1}^{i} \delta_k \mathbf{e}_k + \epsilon \mathbf{e}_i, \mathbf{y}) - V_t^g(\mathbf{x}, \mathbf{y})$ depends only on δ_k, $k = 1, \ldots, i$, $\tilde{y}_m$ and $\mathbf{x}_{[1,i-1]}$, for $\epsilon > 0$, $\delta_k \geq 0$ and $\sum_{k=1}^{i} \delta_k = \epsilon$.*

Proof of Lemma C.1. We will focus on the difference $V_t^g(\mathbf{x}, \mathbf{y} - \sum_{k=1}^{j} \delta_k \mathbf{e}_k + \epsilon \mathbf{e}_j) - V_t^g(\mathbf{x}, \mathbf{y})$ and the other difference satisfies the desired property by symmetry. If we define $\delta_{j+1} = \cdots = \delta_m = 0$ and $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_m)$, the difference can be rewritten as $V_t^g(\mathbf{x}, \mathbf{y} - \boldsymbol{\delta} + \epsilon \mathbf{e}_j) - V_t^g(\mathbf{x}, \mathbf{y})$.

We prove the lemma by induction. Suppose the desired property holds for $t+1$.

If $\tilde{x}_n < \tilde{y}_{j-1}$, there exists $1 \leq j' \leq j-1$ such that $\tilde{y}_{j'-1} \leq \tilde{x}_n < \tilde{y}_{j'}$. Under the greedy matching policy, all the demand and types $1, \ldots, j'-1$ supply is matched, a quantity $\tilde{x}_n - \tilde{y}_{j'-1}$ in type j' supply is matched, and types $j'+1, \ldots, m$ supply will not be matched. This leads to the post-matching levels $\mathbf{Y} = (\mathbf{0}_{[1,j'-1]}, \tilde{y}_{j'} - \tilde{x}_n, \mathbf{y}_{[j'+1,m]})$ for the supply types. Then $V_t^g(\mathbf{x}, \mathbf{y}) = \mathbf{r}^d \mathbf{x}^{\mathsf{T}} + \mathbf{r}^s_{[1,j'-1]} \mathbf{y}^{\mathsf{T}}_{[1,j'-1]} + r^s_{j'}(\tilde{x}_n - \tilde{y}_{j'-1}) + \gamma E V_{t+1}^g(\mathbf{D}, \alpha \mathbf{Y} + \mathbf{S})$.

On the other hand, under the state $(\mathbf{x}, \mathbf{y} - \boldsymbol{\delta} + \epsilon \mathbf{e}_j)$, all the demand will again be fully matched, and the total amounts of demand and supply do not change compared to the state $(\mathbf{x}, \mathbf{y})$. There exists $j' \leq j'' \leq j-1$ such that types $1, \ldots, j''-1$ supply are fully matched and types $j''+1, \ldots, m$ supply are not matched. This leads to the post-matching levels $\mathbf{Y}' = (\mathbf{0}_{[1,j''-1]}, \tilde{y}_{j''} - \tilde{\delta}_{j''} - \tilde{x}_n, \mathbf{y}_{[j''+1,m]} - \boldsymbol{\delta}_{[j''+1,m]}) + \epsilon \mathbf{e}_j$, where $\tilde{\delta}_k \overset{\text{def}}{=} \sum_{\ell=1}^{k} \delta_\ell$ for $1 \leq k \leq m$. Then $V_t^g(\mathbf{x}, \mathbf{y}) = \mathbf{r}^d \mathbf{x}^{\mathsf{T}} + \mathbf{r}^s_{[1,j''-1]}(\mathbf{y}_{[1,j''-1]} - \boldsymbol{\delta}_{[1,j''-1]})^{\mathsf{T}} + r^s_{j''}(\tilde{x}_n - \tilde{y}_{j''-1} + \tilde{\delta}_{j''-1}) + \gamma E V_{t+1}^g(\mathbf{D}, \alpha \mathbf{Y}' + \mathbf{S})$. Let

$$\boldsymbol{\Delta} = \begin{cases} (\mathbf{0}_{[1,j']}, \tilde{y}_{j'} - \tilde{x}_n, \mathbf{y}_{[j'+1,j''-1]}, y_{j''} - (\tilde{y}_{j''} - \tilde{\delta}_{j''} - \tilde{x}_n), \boldsymbol{\delta}_{[j''+1,m]}) & \text{if } j'' > j', \\ (\mathbf{0}_{[1,j'-1]}, \tilde{\delta}_{j'}, \boldsymbol{\delta}_{[j'+1,m]}) & \text{if } j'' = j'. \end{cases}$$

It is easy to verify that $\boldsymbol{\Delta} \geq 0$, $\sum_{k=1}^{m} \Delta_k = \sum_{k=1}^{m} \delta_k = \epsilon$, $\Delta_k = 0$ for $k = j+1, \ldots, m$, and $\boldsymbol{\Delta}$ depends only on ϵ, $\boldsymbol{\delta}$ and $(\tilde{x}_n, \mathbf{y}_{[1,j-1]})$. In addition, $\mathbf{Y}' = \mathbf{Y} - \boldsymbol{\Delta} + \epsilon \mathbf{e}_j$. Then $V_{t+1}^g(\mathbf{D}, \alpha \mathbf{Y} + \mathbf{S}) - V_{t+1}^g(\mathbf{D}, \alpha \mathbf{Y} - \alpha \boldsymbol{\Delta} + \alpha \epsilon \mathbf{e}_j + \mathbf{S})$ depends only on $\boldsymbol{\delta}$, $\tilde{x}_n$, $\tilde{D}_n$ and $\mathbf{y}_{[1,j-1]}$. (Note that $\mathbf{Y}_{[1,j-1]}$ is uniquely determined by $\mathbf{y}_{[1,j-1]}$.) Then the difference

$$\begin{aligned} & V_t^g(\mathbf{x}, \mathbf{y} - \boldsymbol{\delta} + \epsilon \mathbf{e}_j) - V_t^g(\mathbf{x}, \mathbf{y}) \\ = & \mathbf{r}^s_{[1,j'-1]} \mathbf{y}^{\mathsf{T}}_{[1,j'-1]} + r^s_{j'}(\tilde{x}_n - \tilde{y}_{j'-1}) - \mathbf{r}^s_{[1,j''-1]}(\mathbf{y}_{[1,j''-1]} - \boldsymbol{\delta}_{[1,j''-1]})^{\mathsf{T}} - r^s_{j''}(\tilde{x}_n - \tilde{y}_{j''-1} + \tilde{\delta}_{j''-1}) \\ & + \gamma E \left[V_{t+1}^g(\mathbf{D}, \alpha \mathbf{Y} + \mathbf{S}) - V_{t+1}^g(\mathbf{D}, \alpha \mathbf{Y} - \alpha \boldsymbol{\Delta} + \alpha \epsilon \mathbf{e}_j + \mathbf{S}) \right] \end{aligned}$$

depends only on $\boldsymbol{\delta}$, $\tilde{x}_n$ and $\mathbf{y}_{[1,j-1]}$.

If $\tilde{x}_n \geq \tilde{y}_{j-1}$, the greedy matching policy leads to the same post-matching levels under the two states $(\mathbf{x}, \mathbf{y})$ and $(\mathbf{x}, \mathbf{y} - \boldsymbol{\delta} + \epsilon \mathbf{e}_j)$. We see that $V_t^g(\mathbf{x}, \mathbf{y} - \boldsymbol{\delta} + \epsilon \mathbf{e}_j) - V_t^g(\mathbf{x}, \mathbf{y}) = \sum_{k=1}^{j} \delta_k (r^s_j - r^s_k)$, which is independent of $(\mathbf{x}, \mathbf{y})$.

Combining the above analysis, we see that the difference $V_t^g(\mathbf{x},\mathbf{y}-\boldsymbol{\delta}+\epsilon\mathbf{e}_j)-V_t^g(\mathbf{x},\mathbf{y})$ depends only on δ_k, $k=1,\dots,j$, $\tilde{x}_n$ and $\mathbf{y}_{[1,j-1]}$. □

Lemma C.2. *The difference* $V_t^g(\mathbf{x}+\epsilon\mathbf{e}_i^n,\mathbf{y}+\epsilon\mathbf{e}_j^m)-V_t^g(\mathbf{x},\mathbf{y})$ *depends only on* ϵ *and* $(\mathbf{x}_{[1,i-1]},\mathbf{y}_{[1,j-1]},\tilde{x}_n,\tilde{y}_m)$.

Proof of Lemma C.2. First, consider the case $\tilde{x}_n\le\tilde{y}_m$. If $\tilde{x}_n\ge\tilde{y}_{j-1}$, then $V_t^g(\mathbf{x}+\epsilon\mathbf{e}_i^n,\mathbf{y}+\epsilon\mathbf{e}_j^m)-V_t^g(\mathbf{x},\mathbf{y})=(r_i^d+r_j^s)\epsilon$, which is independent of $(\mathbf{x},\mathbf{y})$.

If $\tilde{y}_{j'-1}\le\tilde{x}_n<\tilde{y}_{j'}$ for some $1\le j'\le j-1$, Under state $(\mathbf{x},\mathbf{y})$, the post-matching levels for the supply types are $\mathbf{v}=(\mathbf{0}_{[1,j'-1]},\tilde{y}_{j'}-\tilde{x}_n,\mathbf{y}_{[j'+1,m]})$. The additional amount, ϵ, of type i demand and the extra quantities $\delta_{j'}\overset{\text{def}}{=}\min\{\epsilon,\tilde{y}_{j'}-\tilde{x}_n\}$, $\delta_{j'+1}\overset{\text{def}}{=}\min\{[\epsilon-(\tilde{y}_{j'}-\tilde{x}_n)]^+,y_{j'+1}\}$, $\dots,\delta_{j-1}\overset{\text{def}}{=}\min\{[\epsilon-(\tilde{y}_{j-2}-\tilde{x}_n)]^+,y_{j-1}\}$, $\delta_j\overset{\text{def}}{=}[\epsilon-(\tilde{y}_{j-1}-\tilde{x}_n)]^+$ are matched for supply types $j',j'+1,\dots,j-1,j$, respectively, under the new state $(\mathbf{x}+\epsilon\mathbf{e}_i^n,\mathbf{y}+\epsilon\mathbf{e}_j^m)$. Let $\boldsymbol{\delta}=(0,\dots,0,\delta_{j'},\dots,\delta_j,0,\dots,0)\in\mathbb{R}_+^m$. Note that $\boldsymbol{\delta}$ is a function of $(\tilde{x}_n,\mathbf{y}_{[1,j-1]})$. Then we have

$$V_t^g(\mathbf{x}+\epsilon\mathbf{e}_i^n,\mathbf{y}+\epsilon\mathbf{e}_j^m)-V_t^g(\mathbf{x},\mathbf{y})=r_i^d\epsilon+\sum_{k=1}^m r_k^s\delta_k^s+\gamma E\left[V_{t+1}(\mathbf{D},\alpha(\mathbf{v}-\boldsymbol{\delta}+\epsilon\mathbf{e}_j^m)+\mathbf{S})-V_{t+1}(\mathbf{D},\alpha\mathbf{v}+\mathbf{S})\right].$$

Since $V_{t+1}(\mathbf{D},\alpha(\mathbf{v}-\boldsymbol{\delta}+\epsilon\mathbf{e}_j^m)+\mathbf{S})-V_{t+1}(\mathbf{D},\alpha\mathbf{v}+\mathbf{S})$ depends only on ϵ, $\boldsymbol{\delta}$, $\tilde{D}_n$ and $\mathbf{v}_{[1,j-1]}$ by Lemma C.1, the difference $V_t^g(\mathbf{x}+\epsilon\mathbf{e}_i^n,\mathbf{y}+\epsilon\mathbf{e}_j^m)-V_t^g(\mathbf{x},\mathbf{y})$ depends only on ϵ, $\tilde{x}_n$ and $\mathbf{y}_{[1,j-1]}$, because $\boldsymbol{\delta}$ is defined in terms of ϵ, $\tilde{x}_n$ and $\mathbf{y}_{[1,j-1]}$.

Now consider the case $\tilde{x}_n>\tilde{y}_m$. By symmetry, we can show that $V_t^g(\mathbf{x}+\epsilon\mathbf{e}_i^n,\mathbf{y}+\epsilon\mathbf{e}_j^m)-V_t^g(\mathbf{x},\mathbf{y})$ depends only on ϵ, $\tilde{y}_m$ and $\mathbf{x}_{[1,i-1]}$. Combining those two cases, the difference depends only on ϵ and $(\mathbf{x}_{[1,i-1]},\mathbf{y}_{[1,j-1]},\tilde{x}_n,\tilde{y}_m)$. □

For the one-step-ahead heuristic, in period t, the firm follows the greedy matching from period $t+1$ to the end of the horizon. The firm faces the following optimization problem for the protection level a when matching type i demand with type j supply.

$$\begin{aligned}\max_{a\ge0}\quad f_t(a,\mathbf{x},\mathbf{y})\overset{\text{def}}{=}&-(r_i^d+r_j^s+c+h)a\\&+\gamma EV_{t+1}^g\big(\mathbf{D}_{[1,i-1]},\alpha(\underline{x}_i+a)+D_i,\alpha\mathbf{x}_{[i+1,n]}+\mathbf{D}_{[i+1,n]},\\&\qquad\mathbf{S}_{[1,j-1]},\alpha(\underline{y}_j+a)+S_j,\alpha\mathbf{y}_{[j+1,m]}+\mathbf{S}_{[j+1,m]}\big),\end{aligned}$$

where $\underline{x}_i=(\tilde{x}_i-\tilde{y}_j)^+$ and $\underline{y}_j=(\tilde{y}_j-\tilde{x}_i)^+$.

Since V_{t+1}^g is concave, $f_t(a,\mathbf{x},\mathbf{y})$ is concave in a and the optimal protection level $a^*\in\sup\{a\ge0\mid\lim_{\epsilon\downarrow0}[f(a+\epsilon,\mathbf{x},\mathbf{y})-f(a,\mathbf{x},\mathbf{y})]/\epsilon\ge0\}$. We have

$$f_t(a+\epsilon,\mathbf{x},\mathbf{y})-f_t(a,\mathbf{x},\mathbf{y})$$

$$
\begin{aligned}
&= -(r_i^d + r_j^s + c + h)\epsilon \\
&\quad + \gamma E V_{t+1}^g(\mathbf{D}_{[1,i-1]}, \alpha(\underline{x}_i + a + \epsilon) + D_i, \alpha \mathbf{x}_{[i+1,n]} + \mathbf{D}_{[i+1,n]}, \mathbf{S}_{[1,j-1]}, \alpha(\underline{y}_j + a + \epsilon) + S_j, \alpha \mathbf{y}_{[j+1,m]} + \mathbf{S}_{[j+1,m]}) \\
&\quad - \gamma E V_{t+1}^g(\mathbf{D}_{[1,i-1]}, \alpha(\underline{x}_i + a) + D_i, \alpha \mathbf{x}_{[i+1,n]} + \mathbf{D}_{[i+1,n]}, \mathbf{S}_{[1,j-1]}, \alpha(\underline{y}_j + a) + S_j, \alpha \mathbf{y}_{[j+1,m]} + \mathbf{S}_{[j+1,m]}) \\
&= -(r_i^d + r_j^s + c + h)\epsilon \\
&\quad + \gamma E V_{t+1}^g(\mathbf{D}_{[1,i-1]}, \alpha(\underline{x}_i + a + \epsilon) + D_i, \alpha \mathbf{x}_{[i+1,n]} + \mathbf{D}_{[i+1,n]}, \mathbf{S}_{[1,j-1]}, \alpha(\underline{y}_j + a + \epsilon) + S_j, \alpha \mathbf{y}_{[j+1,m]} + \mathbf{S}_{[j+1,m]}) \\
&\quad - \gamma E V_{t+1}^g(\mathbf{D}_{[1,i-1]}, \alpha(\underline{x}_i + \sum_{i'=i+1}^{n} x_{i'} + a) + D_i, \mathbf{D}_{[i+1,n]}, \mathbf{S}_{[1,j-1]}, \alpha(\underline{y}_j + \sum_{j'=j+1}^{m} y_{j'} + a) + S_j, \mathbf{S}_{[j+1,m]}),
\end{aligned} \tag{14}
$$

where the second equality follows from Lemma C.2. To see this, let $(\mathbf{X}, \mathbf{Y}) = (\mathbf{D}_{[1,i-1]}, \alpha(\underline{x}_i + a) + D_i, \alpha \mathbf{x}_{[i+1,n]} + \mathbf{D}_{[i+1,n]}, \mathbf{S}_{[1,j-1]}, \alpha(\underline{y}_j + a) + S_j, \alpha \mathbf{y}_{[j+1,m]} + \mathbf{S}_{[j+1,m]})$ and $(\mathbf{X}', \mathbf{Y}') = (\mathbf{D}_{[1,i-1]}, \alpha(\underline{x}_i + \sum_{i'=i+1}^{n} x_{i'} + a) + D_i, \mathbf{D}_{[i+1,n]}, \mathbf{S}_{[1,j-1]}, \alpha(\underline{y}_j + \sum_{j'=j+1}^{m} y_{j'} + a) + S_j, \mathbf{S}_{[j+1,m]})$. We see that $\tilde{X}_{[1,i-1]} = \tilde{X}'_{[1,i-1]}$, $\tilde{X}_n = \tilde{X}'_n$, $\tilde{Y}_{[1,j-1]} = \tilde{Y}'_{[1,j-1]}$, $\tilde{Y}_m = \tilde{Y}'_m$. By Lemma C.2, we have $E[V_{t+1}^g(\mathbf{X} + \alpha\epsilon, \mathbf{Y} + \alpha\epsilon) - V_{t+1}^g(\mathbf{X}, \mathbf{Y})] = E[V_{t+1}^g(\mathbf{X}' + \alpha\epsilon, \mathbf{Y}' + \alpha\epsilon) - V_{t+1}^g(\mathbf{X}', \mathbf{Y}')]$, which ensures the second equality in (14).

Let us define

$$
\begin{aligned}
g_t(a, \mathbf{x}, \mathbf{y}) \overset{\text{def}}{=} & -(r_i^d + r_j^s + c + h)a \\
& + \gamma E V_{t+1}^g(\mathbf{D}_{[1,i-1]}, \alpha(\underline{x}_i + \sum_{i'=i+1}^{n} x_{i'} + a) + D_i, \mathbf{D}_{[i+1,n]}, \\
& \qquad \mathbf{S}_{[1,j-1]}, \alpha(\underline{y}_j + \sum_{j'=j+1}^{m} y_{j'} + a) + S_j, \mathbf{S}_{[j+1,m]}).
\end{aligned}
$$

By the above analysis, we have $g_t(a+\epsilon, \mathbf{x}, \mathbf{y}) - g_t(a, \mathbf{x}, \mathbf{y}) = f_t(a+\epsilon, \mathbf{x}, \mathbf{y}) - f_t(a, \mathbf{x}, \mathbf{y})$. Thus it is equivalent to maximize $g_t(a, \mathbf{x}, \mathbf{y})$. By substituting $b = \underline{x}_i + \sum_{i'=i+1}^{n} x_{i'} + a$, we have

$$
\begin{aligned}
g_t(a, \mathbf{x}, \mathbf{y}) = g_t(b, \mathbf{x}, \mathbf{y}) = & -(r_i^d + r_j^s + c + h)(b - \underline{x}_i - \sum_{i'=i+1}^{n} x_{i'}) \\
& + \gamma E V_{t+1}^g(\mathbf{D}_{[1,i-1]}, \alpha b + D_i, \mathbf{D}_{[i+1,n]}, \mathbf{S}_{[1,j-1]}, \alpha(b + \tilde{y}_m - \tilde{x}_n) + S_j, \mathbf{S}_{[j+1,m]}).
\end{aligned} \tag{15}
$$

Note that the above expression depends on $(\mathbf{x}, \mathbf{y})$ only through $\tilde{y}_m - \tilde{x}_n$. By optimizing g_t in terms of b over $b \geq 0$, we have an optimizer $\hat{b}_{ij}(t, \tilde{y}_m - \tilde{x}_n)$ that depends on $\tilde{y}_m - \tilde{x}_n$ and solves $\max_{b \geq 0} g_t(b, \mathbf{x}, \mathbf{y}) = g_t(b, \tilde{y}_m - \tilde{x}_n)$. Since $a \geq 0$, we require $b \geq \underline{x}_i + \sum_{i'=i+1}^{n} x_{i'}$. Thus $b_{ij}^*(t, \tilde{y}_m - \tilde{x}_n) = \hat{b}_{ij}(t, \tilde{y}_m - \tilde{x}_n) \vee (\underline{x}_i + \sum_{i'=i+1}^{n} x_{i'})$ maximizes $g_t(b, \tilde{y}_m - \tilde{x}_n)$ for $b \geq (\underline{x}_i + \sum_{i'=i+1}^{n} x_{i'})$ and $a^* = b_{ij}^*(t, \tilde{y}_m - \tilde{x}_n) - (\underline{x}_i + \sum_{i'=i+1}^{n} x_{i'}) = [\hat{b}_{ij}(t, \tilde{y}_m - \tilde{x}_n) - (\underline{x}_i + \sum_{i'=i+1}^{n} x_{i'})]^+$ solves $\max_{a \geq 0} f_t(a, \mathbf{x}, \mathbf{y})$.

To finish the proof of Theorem C.1, let us define $\tilde{V}_t^g(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \overset{\text{def}}{=} V_t^g(\mathbf{x}, \mathbf{y}) - \mathbf{r}^d \mathbf{x}^\intercal - \mathbf{r}^s \mathbf{x}^\intercal$ (in the same way as we defined $\tilde{V}_t$). Now consider a system (M) with the same parameters expect that we increase both c and h by the same amount M. Let $\tilde{V}_t^M$ denote the value function of this system (M). If M

is sufficiently large, the optimal policy will reduce to the greedy matching policy. Starting from the same initial state, the original system and the new system (M) will have exactly the same trajectory of state because they share the same policy. However, system (M) incurs extra holding and waiting costs. Following this logic, we can infer that $\tilde{V}_t^g(\tilde{\mathbf{x}},\tilde{\mathbf{y}}) = \tilde{V}_t^M(\tilde{\mathbf{x}},\tilde{\mathbf{y}}) - M\sum_{t'=t}^{T}\gamma^{t'-t}E[\alpha^{t'-t}(\tilde{x}_n-\tilde{y}_m)+\sum_{\tau=t+1}^{t'}\alpha^{t'-\tau}(\tilde{D}_n^\tau-\tilde{S}_m^\tau)]^+ - M\sum_{t'=t}^{T}\gamma^{t'-t}E[\alpha^{t'-t}(\tilde{x}_n-\tilde{y}_m)+\sum_{\tau=t+1}^{t'}\alpha^{t'-\tau}(\tilde{D}_n^\tau-\tilde{S}_m^\tau)]^-$, where $\tilde{D}_n^\tau$ and $\tilde{S}_m^\tau$ are the total supply and demand in period τ. Since $\tilde{V}_t^M(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ is $L^\natural$-concave in $(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ by Lemma 7, so is $\tilde{V}_t^g(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$. (Note that the functions $-(z_1-z_2+C)^+$ and $-(z_1-z_2+C)^-$ are $L^\natural$-concave in (z_1,z_2) for any constant C.) Then in period t, the total matching quantity Q^g in the one-step-ahead heuristic under the practice of greedy matching from period $t+1$ will satisfy the same set of properties as in Theorem 7. In terms of the aggregate post-matching level for demand, $\hat{b}_{ij}(t,\tilde{y}_m-\tilde{x}_n)$ is weakly decreasing in $(\tilde{y}_m-\tilde{x}_n)$ and the decreasing rate is dominated by 1. (One can also derive this property from maximizing (15) over $b\geq 0$, with V_t^g rewritten in terms of $\tilde{V}_t^g$.) □

We see that the protection level $a_{ij}^*(t,\mathbf{x},\mathbf{y})$ is determined by the aggregate imbalance between demand and supply levels, $\tilde{y}_m-\tilde{x}_n$, at the beginning of the period and the aggregate post-matching level $\underline{x}_i+\sum_{i'=i+1}^{n}x_{i'}$ of all demand types (i.e., the sum of post-matching levels in all demand types) if type i demand is matched with type j supply as much as possible. Note that $\underline{x}_i+\sum_{i'=i+1}^{n}x_{i'}+a_{ij}^*(t,\mathbf{x},\mathbf{y})$ is the target level for the aggregate post-matching demand. If $\hat{b}_{ij}(t,\tilde{y}_m-\tilde{x}_n)>\underline{x}_i+\sum_{i'=i+1}^{n}x_{i'}$, the target level becomes $\hat{b}_{ij}(t,\tilde{y}_m-\tilde{x}_n)$. Otherwise, it is $\underline{x}_i+\sum_{i'=i+1}^{n}x_{i'}$, which is the outcome of matching type i demand with type j supply to the maximum extent. Therefore, Theorem C.1 implies that the firm's target is to bring the aggregate post-matching demand level as close to $\hat{b}_{ij}(t,\tilde{y}_m-\tilde{x}_n)$ as possible. We also note that the matching of type i demand with type j supply is constrained by the quantity $\overline{x}_i+\sum_{i'=i+1}^{n}x_{i'}$, which is the aggregate level of types $i,i+1,\ldots,n$ demand immediately before the matching of type i demand with type j supply. If $\hat{b}_{ij}(t,\tilde{y}_m-\tilde{x}_n)$ is already equal to or above this level before matching type i demand with type j supply, then these two types will not be matched with each other.

Clearly, greedy matching will be optimal in the last period, i.e., period T, implying that the one-step-ahead heuristic is optimal in the beginning of a two-period model. The following corollary is immediate.

Corollary C.1. *The optimal matching policy in period $T-1$ has the structure as stated in Theorem C.1.*

As an implication, the one-step-ahead heuristic is optimal in the beginning of a two-period model.

D. Vertical Model with Multiplicative Reward Structure

We first provide an example when the reward structure is multiplicative, assortative mating is *not* socially optimal in a two-period matching problem with random demand and supply.

EXAMPLE D.1. Consider the model with the multiplicative reward structure $r_{ij} = r_i^d r_j^s$ for any $i \in \mathcal{D}$ and $j \in \mathcal{S}$. Let both $\mathcal{D}$ and $\mathcal{S}$ contain three types, with $r_1^d = r_1^s = 3$, $r_2^d = r_2^s = 2$, $r_3^d = r_3^s = 0.3$ and $c = h = 0.5$. The total number of periods is $T = 2$. The current state in the beginning of period 1 is $\mathbf{x} = \mathbf{y} = (1,1,1)$. In the next period, $S_1 = 3$ with probability one and $D_i = S_j = 0$ with probability one for all $i \in \mathcal{D}$ and $j \in \mathcal{S} \setminus \{1\}$.

The type 1 supply and the type 1 demand are fully matched in period 1.

If we matching type 2 supply with type 2 demand in period 2, a unit matching reward $r_2^d r_2^s = 4$ will be received. Alternatively, we can retain type 2 demand and match it with type 1 supply in the next period, this will give a unit reward $r_1^s r_2^d = 6$ and incur a unit holding cost $c + h = 1$. (Note that retaining a unit of demand in period 1 means retaining a unit of supply at the same time.) Therefore, preventing type 2 demand from being matched with type 2 supply in period 1 yields a positive payoff of 1.

The maximum unit reward received by matching type 3 demand with type 1 supply is $r_1^s r_3^d = 0.9$, while the unit holding cost for retaining a pair of supply and demand is $c + h = 1 > 0.9$. Since type 2 supply is not matched with type 2 demand in period 1, we will match it with type 3 demand.

The above arguments suggest that the optimal matching decision in period 1 is given by $q_{11}^* = q_{32}^* = 1$ and $q_{ij}^* = 0$ for any $(i,j) \neq (1,1), (3,2)$. Clearly, this decision is *not* assortative mating. $\square$

Next for a special case with one side of the market is lost if not matched, we can characterize the optimal matching policy for a two-period model under the multiplicative reward structure with vertical types. Again, assortative mating is not optimal in the first period. The breakdown of the top-down matching (i.e., assortative mating) is due to the multiplicative reward structure. Such reward structure may result in that holding some high-type supply (or demand) to the next period can be optimal. This is because due to the multiplicative reward structure, saved high-type supply (or demand) can still be profitably matched with low-type demand (or supply) in the next period.

In particular, consider the model with two demand types and two supply types. The number of periods is $T = 2$. Suppose that $\alpha = 0$ and $\beta = 1$. That is, all unmatched demand in the first period is lost, whereas all unmatched supply is carried over to the second period. The reward r_{ij} is increasing in i and j, and is supermodular in (i,j) (i.e., $r_{11} + r_{22} \geq r_{12} + r_{21}$). A special case of this reward structure is the model with vertically differentiated types and the multiplicative reward structure, i.e., $r_{ij} = r_i^d r_j^s$, $r_1^d > r_2^d \geq 0$ and $r_1^s > r_2^s \geq 0$.

Given that $r_{11} + r_{22} \geq r_{12} + r_{21}$ and $r_{11} \geq \max\{r_{12}, r_{21}\}$, we have $(1,1) \succeq (1,2)$ and $(1,1) \succeq (2,1)$. By Theorem 2, this implies that the matching between type 1 demand and type 1 supply is greedy under the optimal policy. Therefore, in period 1 with the state $(\mathbf{x}, \mathbf{y}) = (x_1, x_2, y_1, y_2)$, $q_{11}^* = \min\{x_1, y_1\}$.

• If $x_1 \geq y_1$, then it remains to determine q_{12} and q_{22}. Let $\tilde{x}_1 = x_1 - y_1$.

$$\begin{aligned} V_1(\mathbf{x}, \mathbf{y}) = \max_{q_{12}, q_{22}} \quad & r_{12}q_{12} + r_{22}q_{22} - c(\tilde{x}_1 + x_2 - q_{12} - q_{22}) - h(y_2 - q_{12} - q_{22}) + \gamma E V_2(\mathbf{D}, \mathbf{v} + \mathbf{S}) \\ \text{s.t.} \quad & q_{12} + q_{22} \leq y_2, \\ & 0 \leq q_{12} \leq \tilde{x}_1, \\ & 0 \leq q_{22} \leq x_2, \\ & \mathbf{v} = (0, y_2 - q_{12} - q_{22}). \end{aligned}$$

Due to the assumption that any unmatched type 1 demand will be lost (i.e., $\alpha = 0$), it is easy to see that the arc $(1,2)$ has higher priority over $(2,2)$. There exists two thresholds $\eta_1 \leq \eta_2$ such that it is optimal to reduce type 2 supply as close as possible to the threshold η_1 by matching over the arc $(1,2)$. If there is insufficient type 1 demand to do so, then further reduce type 2 supply as close to the threshold η_2 as much as possible, by matching it with type 2 demand.

• If $x_1 < y_1$, the problem is more complicated. The matching quantities q_{21} and q_{22} remain to be decided. Let $\tilde{y}_1 = y_1 - x_1$.

$$\begin{aligned} V_1(\mathbf{x}, \mathbf{y}) = \max \quad & g(q_{21}, q_{22}, x_2, \tilde{y}_1, y_2) \\ \text{s.t.} \quad & g(q_{21}, q_{22}, x_2, \tilde{y}_1, y_2) = r_{21}q_{21} + r_{22}q_{22} - c(x_2 - q_{21} - q_{22}) - h(\tilde{y}_1 + y_2 - q_{21} - q_{22}) \\ & \qquad\qquad + \gamma E V_2(D_1, D_2, \tilde{y}_1 - q_{21} + S_1, y_2 - q_{22} + S_2) \\ & q_{21} + q_{22} \leq x_2, \\ & 0 \leq q_{21} \leq \tilde{y}_1, \\ & 0 \leq q_{22} \leq y_2. \end{aligned}$$

Due to that any unmatched type 1 supply will be carried over (i.e., $\beta = 1$), $(2,1)$ may not have priority over $(2,2)$. To solve this problem, we first investigate the partial derivatives $\partial g(q_{21}, q_{22}, x_2, \tilde{y}_1, y_2)/\partial q_{21}$ and $\partial g(q_{21}, q_{22}, x_2, \tilde{y}_1, y_2)/\partial q_{22}$. We have

$$\frac{\partial g}{\partial q_{21}} = r_{21} + c + h - \gamma E \frac{\partial V_2(D_1, D_2, \tilde{y}_1 - q_{21} + S_1, y_2 - q_{22} + S_2)}{\partial y_1},$$
$$\frac{\partial g}{\partial q_{22}} = r_{22} + c + h - \gamma E \frac{\partial V_2(D_1, D_2, \tilde{y}_1 - q_{21} + S_1, y_2 - q_{22} + S_2)}{\partial y_2},$$

where the exchange of expectation and partial derivative can be guaranteed by the convexity of V_2 and the monotone convergence theorem. It is easy to see that the optimal policy follows a top-down greedy matching in period 2 and we can calculate the partial derivatives $\frac{\partial V_2(D_1,D_2,\tilde{y}_1-q_{21}+S_1,y_2-q_{22}+S_2)}{\partial y_1}$ and $\frac{\partial V_2(D_1,D_2,\tilde{y}_1-q_{21}+S_1,y_2-q_{22}+S_2)}{\partial y_2}$ in closed form. For example, if $\tilde{y}_1 - q_{21}+S_1+y_2-q_{22}+S_2 < D_1$, then increasing type 2 supply by a small unit results in an increase in the reward in period 2 by $c+r_{12}$ per unit, since one more unit of type 1 demand will be matched. Following this logic,

$$
\begin{aligned}
\frac{\partial g}{\partial q_{22}} =& r_{22}+c+h-\gamma E\left[(c+r_{12})1\left\{\tilde{y}_1+y_2+S_1+S_2-q_{21}-q_{22}<D_1\right\}\right] \\
&-\gamma E\left[(c+r_{22})1\left\{\tilde{y}_1+y_2+S_1+S_2-q_{21}-q_{22}\in(D_1,D_1+D_2)\right\}\right] \\
&+\gamma E\left[h1\left\{\tilde{y}_1+y_2+S_1+S_2-q_{21}-q_{22}>D_1+D_2\right\}\right] \\
=& r_{22}+c+(1+\gamma)h-\gamma(r_{12}-r_{22})F_{S_1+S_2-D_1}(q_{21}+q_{22}-\tilde{y}_1-y_2) \\
&-\gamma(c+h+r_{22})F_{S_1+S_2-D_1-D_2}(q_{21}+q_{22}-\tilde{y}_1-y_2),
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial g}{\partial q_{21}} =& r_{12}+c+h-\gamma E\left[(c+r_{11})1\left\{\tilde{y}_1+y_2-q_{21}-q_{22}+S_1+S_2<D_1\right\}\right] \\
&-\gamma E\left[(r_{11}-r_{12}+r_{22}+c)1\left\{\tilde{y}_1-q_{21}+S_1<D_1, D_1<\tilde{y}_1+y_2-q_{21}-q_{22}+S_1+S_2<D_1+D_2\right\}\right] \\
&-\gamma E\left[(r_{11}-r_{12}-h)1\left\{\tilde{y}_1-q_{21}+S_1<D_1, \tilde{y}_1+y_2-q_{21}-q_{22}+S_1+S_2>D_1+D_2\right\}\right] \\
&-\gamma E\left[(c+r_{21})1\left\{\tilde{y}_1-q_{21}+S_1>D_1, \tilde{y}_1+y_2-q_{21}-q_{22}+S_1+S_2<D_1+D_2\right\}\right] \\
&-\gamma E\left[(r_{21}-r_{22}-h)1\left\{\tilde{y}_1-q_{21}+S_1\in(D_1,D_1+D_2), \tilde{y}_1+y_2-q_{21}-q_{22}+S_1+S_2>D_1+D_2\right\}\right] \\
&+\gamma E\left[h1\left\{\tilde{y}_1-q_{21}+S_1>D_1+D_2\right\}\right] \\
=& r_{12}+c+h-\lambda-\gamma(c+r_{11})F_{S_1+S_2-D_1}(q_{21}+q_{22}-\tilde{y}_1-y_2) \\
&-\gamma(r_{11}-r_{12}+r_{22}+c)E\left\{1\left\{S_1-D_1<q_{21}-\tilde{y}_1, S_1+S_2-D_1-D_2<q_{21}+q_{22}-\tilde{y}_1-y_2\right\}\right\} \\
&+\gamma(r_{11}-r_{12}+r_{22}+c)F_{S_1+S_2-D_1}(q_{21}+q_{22}-\tilde{y}_1-y_2) \\
&-\gamma(r_{11}-r_{12}-h)F_{S_1-D_1}(q_{21}-\tilde{y}_1) \\
&+\gamma(r_{11}-r_{12}-h)E\left\{1\left\{S_1-D_1<q_{21}-\tilde{y}_1, S_1+S_2-D_1-D_2<q_{21}+q_{22}-\tilde{y}_1-y_2\right\}\right\} \\
&-\gamma(c+r_{21})F_{S_1+S_2-D_1-D_2}(q_{21}+q_{22}-\tilde{y}_1-y_2) \\
&+\gamma(c+r_{21})E\left\{1\left\{S_1-D_1<q_{21}-\tilde{y}_1, S_1+S_2-D_1-D_2<q_{21}+q_{22}-\tilde{y}_1-y_2\right\}\right\} \\
&-\gamma(r_{21}-r_{22}-h)F_{S_1-D_1-D_2}(q_{21}-\tilde{y}_1) \\
&-\gamma(r_{21}-r_{22}-h)E\left\{1\left\{S_1-D_1<q_{21}-\tilde{y}_1, S_1+S_2-D_1-D_2<q_{21}+q_{22}-\tilde{y}_1-y_2\right\}\right\}
\end{aligned}
$$

$$+\gamma(r_{21}-r_{22}-h)F_{S_1+S_2-D_1-D_2}(q_{21}+q_{22}-\tilde{y}_1-y_2)+\gamma(r_{21}-r_{22}-h)F_{S_1-D_1}(q_{21}-\tilde{y}_1)$$
$$+\gamma h[1-F_{S_1-D_1-D_2}(q_{21}-\tilde{y}_1)]$$
$$=r_{12}+c+(1+\gamma)h-\gamma(r_{12}-r_{22})F_{S_1+S_2-D_1}(q_{21}+q_{22}-\tilde{y}_1-y_2)-\gamma(r_{11}-r_{12}+r_{22}-r_{21})F_{S_1-D_1}(q_{21}-\tilde{y}_1)$$
$$-\gamma(c+h+r_{22})F_{S_1+S_2-D_1-D_2}(q_{21}+q_{22}-\tilde{y}_1-y_2)-\gamma(r_{21}-r_{22})F_{S_1-D_1-D_2}(q_{21}-\tilde{y}_1)$$

Define

$$H(z)=r_{22}+c+(1+\gamma)h-\gamma(r_{12}-r_{22})F_{S_1+S_2-D_1}(-z)-\gamma(c+h+r_{22})F_{S_1+S_2-D_1-D_2}(-z),$$
$$L(z)=r_{12}-r_{22}-\gamma(r_{11}-r_{12}+r_{22}-r_{21})F_{S_1-D_1}(-z)-\gamma(r_{21}-r_{22})F_{S_1-D_1-D_2}(-z).$$

Both $H(z)$ and $L(z)$ are increasing functions of z, and $\partial g(q_{21},q_{22},x_2,\tilde{y}_1,y_2)/\partial q_{22}=H(\tilde{y}_1+y_2-q_{2\cdot})$, $\partial g(q_{21},q_{22},x_2,\tilde{y}_1,y_2)/\partial q_{21}-\partial g(q_{21},q_{22},x_2,\tilde{y}_1,y_2)/\partial q_{22}=L(\tilde{y}_1-q_{21})$.

We further define θ_H, θ_L, θ_{H+L} and $\theta(y_2)$ as follows.

$$\theta_H=\inf\{z\mid H(z)>0\},$$
$$\theta_L=\inf\{z\mid L(z)>0\},$$
$$\theta_{H+L}=\inf\{z\mid H(z)+L(z)>0\},$$
$$\theta(y_2)=\inf\{z\mid H(y_2+z)+L(z)>0\}.$$

In particular, $H(z)>0$ for all z if $\theta_H=-\infty$, $H(z)\le 0$ if $\theta_H=\infty$ for all z, and $H(\theta_H)=0$ if $-\infty<\theta_H<\infty$. The same thing applies to θ_L, θ_{H+L} and $\theta(y_2)$.

The following lemma provides a comparison of the four thresholds.

Lemma D.1. *Both θ_{H+L} and $\theta(y_2)$ lie between θ_H and θ_L.*

Proof of Lemma D.1. We prove the property for $\theta(y_2)$. The proof for θ_{H+L} is similar.

If $\theta_L\le\theta_H-y_2$, then $H(y_2+\theta_L)+L(\theta_L)\le H(\theta_H)+L(\theta_L)\le 0$ provided that $\theta_L>-\infty$. Thus $\theta(y_2)\ge\theta_L$ if $\theta_L>-\infty$. The same inequality holds trivially if $\theta_L=\infty$. Also, $H(y_2+(\theta_H-y_2))+L(\theta_H-y_2)=H(\theta_H)+L(\theta_H-y_2)\ge H(\theta_H)+L(\theta_L)\ge 0$ provided that $\theta_H<\infty$. Thus, $\theta(y_2)\le\theta_H-y_2$ if $\theta_H<\infty$, and this inequality holds trivially if $\theta=\infty$.

Similarly, we can show that $\theta_H-y_2\le\theta(y_2)\le\theta_L$ if $\theta_L>\theta_H-y_2$. □

Now we are ready to characterize the optimal matching policy in period 1 for the case with $x_1<y_1$.

Proposition D.1 (Optimal matching for 2-period multiplicative reward structure with $\alpha=0$). *Suppose that $x_1<y_1$ and $x_2>0$ in period 1. The optimal policy first matches type 1 demand and* *type 1 supply in a greedy fashion. For the matching over the arcs* $(2,1)$ *and* $(2,2)$*, the optimal policy first matches type 2 demand and type 1 supply, until the quantity of type 1 supply reduces to the threshold* θ_L *or as close to as possible (i.e.,* $q^*_{21} = (\tilde{y}_1 - \theta_L)^+ \wedge \tilde{y}_1$*, so that type 1 supply reduces to the closest amount to* θ_L *within the interval* $[0, \tilde{y}_1]$*). After that, further match type 2 demand and type 2 supply until either the total quantity of supply (both type 1 and type 2) reduces to* θ_H *or all type 2 demand is exhausted, if it is possible to do so. If, however, either the total supply is already below* θ_H *before the matching between type 2 demand and type 2 supply, or all type 2 demand is exhausted by the matching over* $(2,1)$*, then set the optimal matching quantity* q^*_{22} *to zero, and revise the matching quantity* q^*_{21} *so that type 1 would reach the threshold* $\theta(y_2)$ *or as close to as possible. If the total supply is still above* θ_H *and type 2 demand has not exhausted even if we fully match type 2 supply (with type 2 demand), then set* $q^*_{22} = y_2$*, and revise* q^*_{21} *so that type 1 supply would reach the threshold* θ_{H+L} *or as close to as possible.*

Mathematically, Proposition D.1 can be described as follows.

If $(\tilde{y}_1 - \theta_L)^+ \wedge \tilde{y}_1 \le (\tilde{y}_1 + y_2 - \theta_H) \wedge x_2 \le (\tilde{y}_1 - \theta_L)^+ \wedge \tilde{y}_1 + y_2$, then $q^*_{21} = (\tilde{y}_1 - \theta_L)^+ \wedge \tilde{y}_1$ and $q^*_{22} = (\tilde{y}_1 + y_2 - \theta_H) \wedge x_2 - (\tilde{y}_1 - \theta_L)^+ \wedge \tilde{y}_1$. In this case, the matching quantity $q^*_{21} = (\tilde{y}_1 - \theta_L)^+ \wedge \tilde{y}_1$ reduces type 1 supply to θ_L or as close to as possible (i.e., to 0 if $\theta_L < 0$ and to $\tilde{y}_1$ if $\theta_L > \tilde{y}_1$). Then, the matching quantity $q^*_{22} = (\tilde{y} + y_2 - \theta_H) \wedge x_2 - (\tilde{y}_1 - \theta_L)^+ \wedge \tilde{y}_1$ further reduces the total supply to θ_H or as close to as possible (i.e., type 2 demand will be exhausted if x_2 is insufficient to reduce the total supply to θ_H).

If $(\tilde{y}_1 + y_2 - \theta_H) \wedge x_2 < (\tilde{y}_1 - \theta_L)^+ \wedge \tilde{y}_1$, then $q^*_{21} = [\tilde{y}_1 - \theta(y_2)]^+ \wedge \tilde{y}_1 \wedge x_2$ and $q^*_{22} = 0$. In this case, the goal of reducing type 1 supply to θ_L or as close to as possible by matching it with type 2 demand will cause either the total supply to fall below θ_H (when $(\tilde{y}_1 - \theta_L)^+ \wedge \tilde{y}_1 \le \tilde{y}_1 + y_2 - \theta_H$) or all the type 2 supply exhausted (when $(\tilde{y}_1 - \theta_L)^+ \wedge \tilde{y}_1 \le x_2$). Then, we do not match type 2 demand with type 2 supply, and revise the matching between type 2 demand and type 1 supply so that type 1 supply reduces to $\theta(y_2)$ or as close to as possible (i.e., to the closest value to $\theta(y_2)$ within the interval $[0, \tilde{y}_1]$ if there is sufficient type 2 demand, or exhaust all the type 2 demand on hand otherwise).

(Note that when $(\tilde{y}_1 + y_2 - \theta_H) \wedge x_2 = (\tilde{y}_1 - \theta_L)^+ \wedge \tilde{y}_1$, one can verify that the quantity $(\tilde{y}_1 - \theta_L)^+ \wedge \tilde{y}_1$ coincides with $[\tilde{y}_1 - \theta(y_2)]^+ \wedge \tilde{y}_1 \wedge x_2$. In other words, both the formulas in the first case and in the second case gives the same matching plan. Thus, when the matching quantity $(\tilde{y}_1 - \theta_L)^+ \wedge \tilde{y}_1$ precisely exhausts x_2, revising the matching over (2,1) would not change the decision.)

If $(\tilde{y}_1 + y_2 - \theta_H) \wedge x_2 > (\tilde{y}_1 - \theta_L)^+ \wedge \tilde{y}_1 + y_2$, then $q^*_{21} = (\tilde{y}_1 - \theta_{H+L})^+ \wedge \tilde{y}_1 \wedge (x_2 - y_2)$ and $q^*_{22} = y_2$. In this case, we first let type 1 supply reduce to the closest amount to θ_L within the interval $[0, \tilde{y}_1]$

by matching it with type 2 demand. After that, however, the total supply will still be above θ_H and type 2 demand will not be exhausted even if we fully exhaust type 2 supply. We then fully match type 2 supply (with type 2 demand), and revise the matching between type 2 demand and type 1 supply so that type 1 supply reduces to θ_{H+L} or as close to as possible.

Proof of Proposition D.1. By applying the KKT conditions, it is easy to derive the following optimality conditions.

For (q_{21}, q_{22}) to be optimal, there must exist $\lambda \geq 0$ such that

(i) $\lambda(q_{21}+q_{22}-x_2)=0$;

(ii) $\partial g(q_{21}, q_{22}, x_2, \tilde{y}_1, y_2)/\partial q_{21} - \lambda = 0$ and $q_{21} \in (0, \tilde{y}_1)$; or $\partial g(q_{21}, q_{22}, x_2, \tilde{y}_1, y_2)/\partial q_{21} - \lambda \leq 0$ and $q_{21}=0$; or $\partial g(q_{21}, q_{22}, x_2, \tilde{y}_1, y_2)/\partial q_{21} - \lambda \geq 0$ and $q_{21}=\tilde{y}_1$;

(iii) $\partial g(q_{21}, q_{22}, x_2, \tilde{y}_1, y_2)/\partial q_{22} - \lambda = 0$ and $q_{22} \in (0, y_2)$; or $\partial g(q_{21}, q_{22}, x_2, \tilde{y}_1, y_2)/\partial q_{22} - \lambda \leq 0$ and $q_{22}=0$; or $\partial g(q_{21}, q_{22}, x_2, \tilde{y}_1, y_2)/\partial q_{22} - \lambda \geq 0$ and $q_{22}=y_2$;

(iv) $q_{21}+q_{22} \leq x_2$.

Since the optimization problem is a convex program, the above conditions are sufficient and necessary. In the followings, we show that the matching decision (q_{21}^*, q_{22}^*) described in the proposition will satisfy those conditions.

Case 1. $(\tilde{y}_1-\theta_L)^+ \wedge \tilde{y}_1 \leq (\tilde{y}_1+y_2-\theta_H)\wedge x_2 \leq (\tilde{y}_1-\theta_L)^+ \wedge \tilde{y}_1 + y_2$. We let $q_{21}^* = (\tilde{y}_1-\theta_L)^+ \wedge \tilde{y}_1$ and $q_{22}^* = (\tilde{y}+y_2-\theta_H)\wedge x_2 - (\tilde{y}_1-\theta_L)^+ \wedge \tilde{y}_1$.

Let $\lambda = H(\tilde{y}_1+y_2-(\tilde{y}_1+y_2-\theta_H)\wedge x_2) \geq H(\theta_H) \geq 0$.

Then, $\partial g(q_{21}^*, q_{22}^*, x_2, \tilde{y}_1, y_2)/\partial q_{21} = H(\tilde{y}_1+y_2-(\tilde{y}_1+y_2-\theta_H)\wedge x_2) + L(\tilde{y}_1-(\tilde{y}_1-\theta_L)^+\wedge\tilde{y}_1) - \lambda = L(\tilde{y}_1-(\tilde{y}_1-\theta_L)^+\wedge\tilde{y}_1)$, which equals $L(\theta_L)$ if $0<\theta_L<\tilde{y}_1$ (such that $q_{21}^* \in (0,\tilde{y}_1)$), equals $L(0) \geq L(\theta_L) \geq 0$ if $\theta_L \leq 0$ (such that $q_{21}^* = \tilde{y}_1$), and equals $L(\tilde{y}_1) \leq L(\theta_L) \leq 0$ if $\theta_L \geq \tilde{y}_1$ (such that $q_{21}^*=0$).

Also, $\partial g(q_{21}^*, q_{22}^*, x_2, \tilde{y}_1, y_2)/\partial q_{22} = H(\tilde{y}_1+y_2-(\tilde{y}_1+y_2-\theta_H)\wedge x_2) - \lambda = 0$.

Moreover, if $q_{21}^*+q_{22}^* = (\tilde{y}_1+y_2-\theta_H)\wedge x_2 < x_2$, then $\tilde{y}_1+y_2-\theta_H < x_2$, implying that $\lambda = H(\theta_H) = 0$. Thus $\lambda(q_{21}^*+q_{22}^*-x_2)=0$.

The optimality conditions are satisfied.

Case 2. $(\tilde{y}_1+y_2-\theta_H)\wedge x_2 < (\tilde{y}_1-\theta_L)^+ \wedge \tilde{y}_1$.

Let $q_{21}^* = [\tilde{y}_1-\theta(y_2)]^+ \wedge \tilde{y}_1 \wedge x_2$ and $q_{22}^*=0$.

First, consider the case $x_2 \leq \tilde{y}_1+y_2-\theta_H$. We have $(\tilde{y}_1-\theta_L)^+\wedge\tilde{y}_1 \geq x_2 > 0$, implying that $\tilde{y}_1 > \theta_L$ and $\tilde{y}_1 \geq x_2$. Since $\theta(y_2)$ is between $\theta_H - y_2$ and θ_L, $\tilde{y}_1-\theta(y_2)$ is between $\tilde{y}_1-\theta_L$ and $\tilde{y}_1+y_2-\theta_H$, both greater than or equal to x_2. Thus $q_{21}^* = [\tilde{y}_1-\theta(y_2)]^+\wedge\tilde{y}_1\wedge x_2 = \tilde{y}_1\wedge x_2 = x_2$ and $q_{22}^*=0$. Then, $H(\tilde{y}_1+y_2-x_2)+L(\tilde{y}_1-x_2) \geq H(y_2+\theta(y_2))+L(\theta(y_2)) \geq 0$ and $L(\tilde{y}_1-x_2) \geq L(\theta_L) \geq 0$. Let $\lambda = H(\tilde{y}_1+y_2-x_2)+L(\tilde{y}_1-x_2) \geq H(y_2+\theta(y_2))+L(\theta(y_2))$. We have $\partial g(q_{21}^*, q_{22}^*, x_2, \tilde{y}_1, y_2)/\partial q_{21} - \lambda =$

$H(\tilde{y}_1+y_2-x_2)+L(\tilde{y}_1-x_2)\geq H(y_2+\theta(y_2))+L(\theta(y_2))-\lambda=0$ and $\partial g(q_{21}^*,q_{22}^*,x_2,\tilde{y}_1,y_2)/\partial q_{22}-\lambda=-L(\tilde{y}_1-x_2)\leq 0$. Thus, $(q_{21}^*,q_{22}^*)=(x_2,0)$ is optimal.

Next, consider the case $x_2>\tilde{y}_1+y_2-\theta_H$.

If $\theta_H-y_2\geq\theta_L$, then $\theta_H-y_2\geq\theta(y_2)\geq\theta_L$. If $\theta(y_2)\in[0,\tilde{y}_1]$, then $q_{21}^*=[\tilde{y}_1-\theta(y_2)]\wedge x_2$. Let $\lambda=H(\tilde{y}_1+y_2-[\tilde{y}_1-\theta(y_2)]\wedge x_2)+L(\tilde{y}_1-[\tilde{y}_1-\theta(y_2)]\wedge x_2)\geq H(\tilde{y}_1+y_2-[\tilde{y}_1-\theta(y_2)])+L(\tilde{y}_1-[\tilde{y}_1-\theta(y_2)])=H(y_2+\theta(y_2))+L(\theta(y_2))=0$. Then, $\partial g(q_{21}^*,q_{22}^*,x_2,\tilde{y}_1,y_2)/\partial q_{21}-\lambda=H(\tilde{y}_1+y_2-[\tilde{y}_1-\theta(y_2)]\wedge x_2)+L(\tilde{y}_1-[\tilde{y}_1-\theta(y_2)]\wedge x_2)-\lambda=0$ and $\partial g(q_{21}^*,q_{22}^*,x_2,\tilde{y}_1,y_2)/\partial q_{22}-\lambda=H(\tilde{y}_1+y_2-[\tilde{y}_1-\theta(y_2)]\wedge x_2)-\lambda=-L(\tilde{y}_1-[\tilde{y}_1-\theta(y_2)]\wedge x_2)\leq -L(\tilde{y}_1-[\tilde{y}_1-\theta(y_2)])=L(\theta(y_2))\leq L(\theta_L)\leq 0$. Moreover, when $q_{21}^*+q_{22}^*=[\tilde{y}_1-\theta(y_2)]\wedge x_2<x_2$, we have $q_{21}^*=\tilde{y}_1-\theta(y_2)<x_2$, implying that $\lambda=H(y_2+\theta(y_2))+L(\theta(y_2))=0$. Thus the optimality conditions are satisfied.

If $\theta_H-y_2\geq\theta(y_2)\geq\theta_L$ and $\theta(y_2)>\tilde{y}_1$, then $q_{21}^*=0$. We have $\partial g(q_{21}^*,q_{22}^*,x_2,\tilde{y}_1,y_2)/\partial q_{21}=H(\tilde{y}_1+y_2)+L(\tilde{y}_1)\leq H(y_2+\theta(y_2))+L(\theta(y_2))\leq 0$ and $\partial g(q_{21}^*,q_{22}^*,x_2,\tilde{y}_1,y_2)/\partial q_{22}=H(\tilde{y}_1+y_2)\leq H(y_2+\theta(y_2))\leq H(\theta_H)\leq 0$. Let $\lambda=0$ and the optimality conditions are satisfied.

If $\theta_H-y_2\geq\theta(y_2)\geq\theta_L$ and $\theta(y_2)<0$, then $q_{21}^*=\tilde{y}_1\wedge x_2$. In the case where $\tilde{y}_1<x_2$, we have $q_{21}^*=\tilde{y}_1$. Then $\partial g(q_{21}^*,q_{22}^*,x_2,\tilde{y}_1,y_2)/\partial q_{21}=H(y_2)+L(0)\geq H(y_2+\theta(y_2))+L(\theta(y_2))\geq 0$. The condition $(\tilde{y}_1+y_2-\theta_H)\wedge x_2<(\tilde{y}_1-\theta_L)^+\wedge\tilde{y}_1=\tilde{y}_1$ implies that $\tilde{y}_1+y_2-\theta_H<\tilde{y}_1$, since $\tilde{y}_1<x_2$. Thus $y_2<\theta_H$ and $\partial g(q_{21}^*,q_{22}^*,x_2,\tilde{y}_1,y_2)/\partial q_{22}=H(y_2)\leq H(\theta_H)\leq 0$. Thus, $\lambda=0$ will satisfy the optimality conditions. In the case $\tilde{y}_1\geq x_2$, $q_{21}^*=x_2$. Let $\lambda=H(\tilde{y}_1+y_2-x_2)+L(\tilde{y}_1-x_2)$. Since $\tilde{y}_1-x_2\geq 0>\theta(y_2)$, we have $\lambda\geq H(y_2+\theta(y_2))+L(\theta(y_2))\geq 0$ and $L(\tilde{y}_1-x_2)\geq L(\theta(y_2))\geq L(\theta_L)\geq 0$. It follows that $\partial g(q_{21}^*,q_{22}^*,x_2,\tilde{y}_1,y_2)/\partial q_{21}-\lambda=0$ and $\partial g(q_{21}^*,q_{22}^*,x_2,\tilde{y}_1,y_2)/\partial q_{22}-\lambda=-L(\tilde{y}_1-x_2)\leq 0$. The optimality conditions are satisfied.

If $\theta_H-y_2\leq\theta(y_2)\leq\theta_L$, then $\tilde{y}_1-\theta_L\leq\tilde{y}_1-\theta(y_2)\leq\tilde{y}_1+y_2-\theta$. The condition $(\tilde{y}_1+y_2-\theta_H)\wedge x_2<(\tilde{y}_1-\theta_L)^+\wedge\tilde{y}_1$ leads to $(\tilde{y}_1-\theta_L)^+>\tilde{y}_1+y_2-\theta_H\geq\tilde{y}_1-\theta(y_2)\geq\tilde{y}_1-\theta_L$. This implies that $\tilde{y}_1-\theta_L<0$, so that $\tilde{y}_1-\theta(y_2)\leq(\tilde{y}_1-\theta_L)^+=0$ and $\tilde{y}_1+y_2-\theta_H\leq(\tilde{y}_1-\theta_L)^+=0$. Therefore, $q_{21}^*=(\tilde{y}_1-\theta(y_2))\wedge\tilde{y}_1\wedge x_2$. We then have $\partial g(q_{21}^*,q_{22}^*,x_2,\tilde{y}_1,y_2)/\partial q_{21}=H(\tilde{y}_1+y_2)+L(\tilde{y}_1)\leq H(y_2+\theta(y_2))+L(\theta(y_2))\leq 0$ and $\partial g(q_{21}^*,q_{22}^*,x_2,\tilde{y}_1,y_2)/\partial q_{22}=H(\tilde{y}_1+y_2)\leq H(\theta_H)\leq 0$. Thus, $\lambda=0$ satisfies the optimality conditions.

Case 3. $(\tilde{y}_1+y_2-\theta_H)\wedge x_2>(\tilde{y}_1-\theta_L)^+\wedge\tilde{y}_1+y_2$.

Let $q_{21}^*=(\tilde{y}_1-\theta_{H+L})^+\wedge\tilde{y}_1\wedge(x_2-y_2)$ and $q_{22}^*=y_2$. Note that the condition $(\tilde{y}_1+y_2-\theta_H)\wedge x_2>(\tilde{y}_1-\theta_L)^+\wedge\tilde{y}_1+y_2$ ensures that $x_2>y_2$ and $\theta_H<\infty$.

If $\theta_L>0$, then $\tilde{y}_1+y_2-\theta_H>(\tilde{y}_1-\theta_L)^+\wedge\tilde{y}_1+y_2=(\tilde{y}_1-\theta_L)^++y_2\geq\tilde{y}_1-\theta_L+y_2$, or equivalently, $\theta_H<\theta_L$. It follows that $\theta_H\leq\theta_{H+L}\leq\theta_L$. If further $\tilde{y}_1\leq\theta_{H+L}$, then $q_{21}^*=0$ and $(\tilde{y}_1+y_2-\theta_H)\wedge x_2\geq(\tilde{y}_1-\theta_L)^++y_2=y_2$. (Note that $0\leq(\tilde{y}_1-\theta_L)^+\leq(\tilde{y}_1-\theta_{H+L})^+=0$.)

Thus, $\tilde{y}_1 \geq \theta_H$. We have $\partial g(q_{21}^*, q_{22}^*, x_2, \tilde{y}_1, y_2)/\partial q_{21} = H(\tilde{y}_1) + L(\tilde{y}_1) \leq H(\theta_{H+L}) + L(\theta_{H+L}) \leq 0$ and $\partial g(q_{21}^*, q_{22}^*, x_2, \tilde{y}_1, y_2)/\partial q_{22} = H(\tilde{y}_1) \geq H(\theta_H) \geq 0$. $\lambda = 0$ satisfies the optimality conditions.

If $\theta_L > 0$ and $\tilde{y}_1 > \theta_{H+L}$, then $q_{21}^* = (\tilde{y}_1 - \theta_{H+L}) \wedge \tilde{y}_1 \wedge (x_2 - y_2)$. In the case where $\tilde{y}_1 - \theta_{H+L} \leq \tilde{y}_1 \wedge (x_2 - y_2)$, we have $q_{21}^* = \tilde{y}_1 - \theta_{H+L}$ and $\partial g(q_{21}^*, q_{22}^*, x_2, \tilde{y}_1, y_2)/\partial q_{21} = H(\theta_{H+L}) + L(\theta_{H+L}) = 0$. Moreover, $\partial g(q_{21}^*, q_{22}^*, x_2, \tilde{y}_1, y_2)/\partial q_{22} = H(\theta_{H+L}) \geq H(\theta_H) \geq 0$. Let $\lambda = 0$ and the optimality conditions are satisfied. In the case where $\tilde{y}_1 \leq (\tilde{y}_1 - \theta_{H+L}) \wedge (x_2 - y_2)$, we have $\theta_{H+L} \leq 0$ and $q_{21}^* = \tilde{y}_1$. $\partial g(q_{21}^*, q_{22}^*, x_2, \tilde{y}_1, y_2)/\partial q_{21} = H(0) + L(0) \geq H(\theta_{H+L}) + L(\theta_{H+L}) \geq 0$ and $\partial g(q_{21}^*, q_{22}^*, x_2, \tilde{y}_1, y_2)/\partial q_{22} = H(0) \geq H(\theta_{H+L}) \geq H(\theta_H) \geq 0$. Let $\lambda = 0$ and again the optimality conditions are satisfied. In the case where $x_2 - y_2 \leq (\tilde{y}_1 - \theta_{H+L}) \wedge \tilde{y}_1$, we have $\tilde{y}_1 + y_2 - x_2 \geq \theta_{H+L}$ and $q_{21}^* = x_2 - y_2$. We have $H(\tilde{y}_1 + y_2 - x_2) + L(\tilde{y}_1 + y_2 - x_2) \geq H(\theta_{H+L}) + L(\theta_{H+L}) \geq 0$ and $L(\tilde{y}_1 + y_2 - x_2) \leq L(\theta_L) \leq 0$, where the latter holds because of the condition $(\tilde{y}_1 + y_2 - \theta_H) \wedge x_2 > (\tilde{y}_1 - \theta_L)^+ \wedge \tilde{y}_1 + y_2$, implying $x_2 > \tilde{y}_1 - \theta_L + y_2$. Then, $\partial g(q_{21}^*, q_{22}^*, x_2, \tilde{y}_1, y_2)/\partial q_{21} - \lambda = H(\tilde{y}_1 + y_2 - x_2) + L(\tilde{y}_1 + y_2 - x_2) - \lambda = 0$ and $\partial g(q_{21}^*, q_{22}^*, x_2, \tilde{y}_1, y_2)/\partial q_{21} - \lambda = -L(\tilde{y}_1 + y_2 - x_2) \geq 0$.

If $\theta_L \leq 0$, then $(\tilde{y}_1 + y_2 - \theta_H) \wedge x_2 > \tilde{y}_1 + y_2$, which implies $\theta_H < 0$ and $\tilde{y}_1 < x_2 - y_2$. Since θ_{H+L} lies between θ_H and θ_L, $\theta_{H+L} \leq 0$. Therefore, $q_{21}^* = \tilde{y}_1 \wedge (x_2 - y_2) = \tilde{y}_1$. Thus $\partial g(q_{21}^*, q_{22}^*, x_2, \tilde{y}_1, y_2)/\partial q_{21} = H(0) + L(0) \geq H(\theta_{H+L}) + L(\theta_{H+L}) \geq 0$ and $\partial g(q_{21}^*, q_{22}^*, x_2, \tilde{y}_1, y_2)/\partial q_{22} = H(0) \geq H(\theta_H) \geq 0$. Let $\lambda = 0$ and the optimality conditions are satisfied. □

E. Numerical Study of the Deterministic Heuristic

We now test the effectiveness of the deterministic heuristic. Consider a 10-period dynamic matching problem with 5 supply types and 5 demand types. For each instance of the problem, we generate the parameters uniformly at random as follows.

Let $r_{ij} \sim \text{Uniform}[50, 150]$ (for all $i \in \mathcal{D}$ and $j \in \mathcal{S}$), $c \sim \text{Uniform}[0, 50]$, $h \sim \text{Uniform}[0, 50]$, $\alpha \sim \text{Uniform}[0, 1]$, $\beta \sim \text{Uniform}[0, 1]$, $\mu = ES \sim \text{Uniform}[10, 25]$, $\lambda = ED \sim \text{Uniform}[10, 25]$, $\gamma \sim \text{Uniform}[0.8, 1]$.

In addition, we also randomly generate the initial state $(\mathbf{x}^0, \mathbf{y}^0)$ at the beginning of the first period. We let $x_i^0 \sim \text{Uniform}[0, 30]$ and $y_j^0 \sim \text{Uniform}[0, 30]$ for all $i \in \mathcal{D}$ and $j \in \mathcal{S}$.

We run two sets of numerical experiments as described as follows.

(a) Demand and supply follow a uniform distribution. For given realizations of λ_i and μ_j, we generate $\delta_i^d \sim \text{Uniform}[0, \lambda_i]$ and $\delta_j^s \sim \text{Uniform}[0, \mu_j]$. Then, we let $D_i \sim \text{Uniform}[\lambda_i - \delta_i^d, \lambda_i + \delta_i^d]$ and $S_j \sim \text{Uniform}[\mu_j - \delta_j^s, \mu_j + \delta_j^s]$.

(b) Demand and supply follow a normal distribution. For given realizations of λ_i and μ_j, we generate $\sigma_i^d \sim \text{Uniform}[0, \lambda_i/3]$ and $\sigma_j^s \sim \text{Uniform}[0, \mu_j/3]$. Then, we let $D_i \sim \text{Normal}(\lambda_i, \sigma_i^d)$ and $S_j \sim \text{Normal}(\mu_j, \sigma_j^s)$.

Note that all the parameters are generated independently. For each randomly generated instance, we solve the 10-period deterministic problem (P) and obtain the optimal value V^{det}, which is an upper bound of the optimal value V^{opt} of the stochastic problem. Let $\tilde{V}$ be the optimal value of the expected total discounted reward minus costs, when the deterministic heuristic is applied throughout the decision horizon. We calculate $\tilde{V}$ approximately by simulation: For each randomly generated sample path ω, in period t ($t = 1, \ldots, T$) with state $(\mathbf{x}_t(\omega), \mathbf{y}_t(\omega))$, apply the optimal decision from solving the $(T - t + 1)$-period problem with initial state $(\mathbf{x}_t(\omega), \mathbf{y}_t(\omega))$; The total reward minus cost for the sample path ω can be easily calculated; Then we average over 5000 sample paths to obtain the approximate value of $\tilde{V}$. Since $(V^{\text{opt}} - \tilde{V})/V^{\text{opt}} \leq (V^{\text{det}} - \tilde{V})/V^{\text{det}}$, the relative error by the deterministic heuristic is small as long as the right-hand-side of the above inequality is small. Thus we focus on $\rho = (V^{\text{det}} - \tilde{V})/V^{\text{det}}$ to measure the relative error.

For set (a) of the experiments, 600 instances are generated. Among the 600 instances, the maximum value of ρ is 21.24%, the mean is 9.84% and the median is 9.51%.

For set (b) of the experiments, 820 instances are generated. Among the 820 instances, the maximum value of ρ is 19.24%, the mean is 7.24% and the median is 6.79%.

The empirical cumulative distribution functions for the two sets of ρ values are shown in Figures 8 and 9, respectively. We see that the values of ρ are relatively small. Since ρ is just an upper bound of the relative error, the relative error would be even smaller.

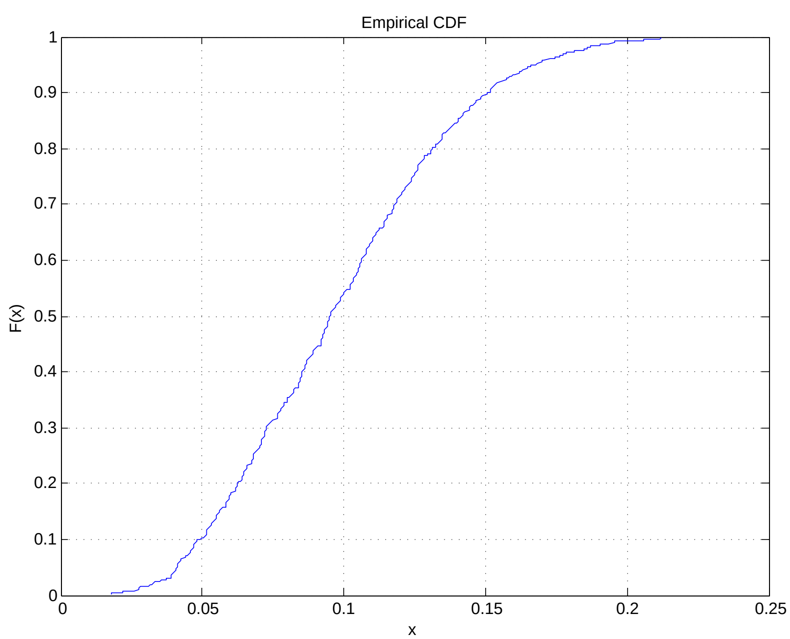

Figure 8 Empirical cdf of ρ: Uniformly distributed demand and supply

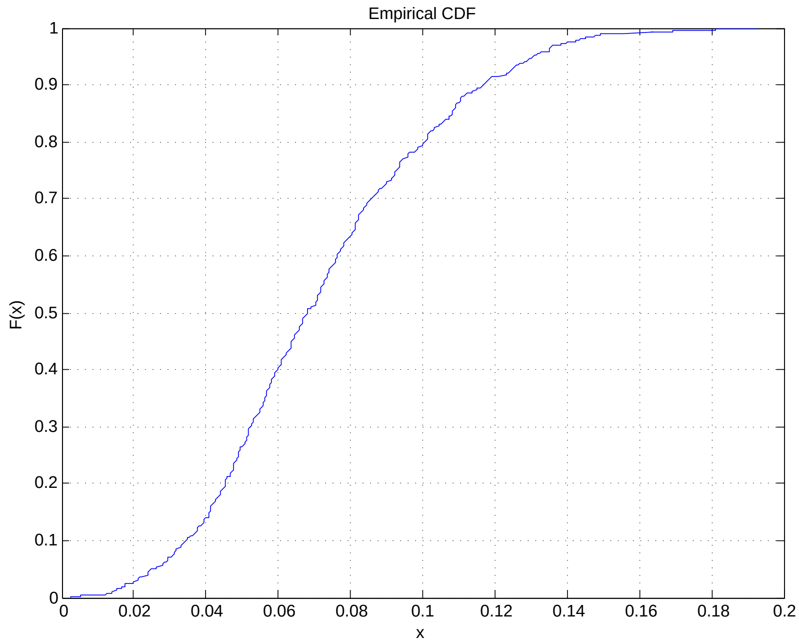

Figure 9 Empirical cdf of ρ: Normally distributed demand and supply