Managing Inventory for Entrepreneurial Firms with Trade Credit and Payment Defaults

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This paper considers an entrepreneurial firm that periodically orders inventory to satisfy demand in a finite horizon. The firm offers trade credit to its customer while receiving one from its supplier. In addition to standard inventory-related costs, cash-related costs are incurred in each period. The cash-related cost is composed of a default penalty cost due to a cash shortage and an interest gain (negative cost) due to excess cash after inventory payments. The objective is to obtain an inventory policy that maximizes the working capital at the end of the horizon. We show that this problem is equivalent to that of minimizing the total inventory related cost and the cash-related cost in the horizon. The model with ample cash reduces to the traditional inventory model. For the general model, we prove that a state-dependent policy is optimal. To facilitate implementation and reveal insights, we consider a simplified model in which a myopic policy is optimal. A numerical study suggests that the myopic policy is effective for the original system. The myopic policy generalizes the classic base-stock policy and resembles practical working capital management under which firms make inventory decisions according to their working capital status. The policy parameters have closed-form expressions, which show the impact of demand and cost parameters on the inventory decision. Our study assesses the value of considering financial flows when the firm makes the inventory decision and reveals insights that are consistent with empirical findings in the literature.

Key words: trade credit, inventory policy, payment default, working capital

1 Introduction

Small and medium-sized enterprises (SMEs) account for 95% to 99% of registered business depending on the country. The contribution of SMEs to GDP ranges from less than 20% in poor countries to about half of GDP in rich economies (International Trade Centre 2009). Trade credit finance is the lifeline for small or entrepreneurial firms as they are newly established and usually have difficulty securing bank loans (Klonowski 2014, p.89). This is particularly true during the financial crisis when banks are extremely risk-averse. Managing inventory with trade credit is part of working capital management for firms. Working capital refers to the difference between current assets and current liabilities (short-term assets and liabilities with maturities of less than one year). In the balance sheet, current assets include cash, inventory, and accounts receivable (A/R). Current liabilities include accounts payable (A/P) and short-term loans. With trade credit, inventory decisions directly affect working capital levels as the deferred inventory payment and the delayed sales collection are recorded as A/P and A/R, respectively. Working capital represents liquidity and financial viability for SMEs or entrepreneurial firms. Thus, it is very important to investigate how entrepreneurial firms should maximize their working capital when trade credit is present in their business transactions.

We consider an entrepreneurial firm in the middle of a supply chain. The firm periodically orders inventory from its supplier to fulfill stochastic demand received from its customer in a finite horizon. The firm has no credit history to secure bank loans, so trade credit is the single source of external financing. Specifically, the firm offers trade credit to its customer while receiving one from its supplier. The trade credit is a one-part (net term) contract, that is, the payment is due within a certain time period after the invoice is issued. The firm pays for the ordered inventory after a deferral *payment period* following the delivery of goods, and receives sales revenue after a *collection period* following the demand. As suggested by the 1998 National Survey of Small Business Finances (NSSBF) data, a big portion of the small firms declared that they had made some payments to their suppliers after the due date of the trade credit. These post due-date payments, referred to as payment defaults, often incur monetary penalties for the buyers (see discussions in §2.4.2). In light of this, we introduce a default penalty cost incurred upon the unfulfilled payment to the firm's supplier. On the other hand, there is a positive interest gain for the excess cash after inventory payments. This represents the fact that the firm has a basic investment function.

Most inventory models in the literature do not explicitly consider the interaction between inventory decisions and cash flows under the assumption of ample cash supply. We believe that it is important to study this connection for the following two reasons. First, during the financial crisis, it is difficult for entrepreneurial firms to secure sufficient cash to meet their short-term operations. Consequently, the inventory decision plays an important role on a firm's liquidity and operational efficiency. More specifically, a current inventory order will affect its future cash payment. If a firm orders too much, it not only incurs a higher holding cost, but also increases the chance of future payment defaults. On the other hand, if a firm orders too little, it is more likely to stock out, although a higher return of interest gains is expected. Thus, there are clear tradeoffs between these financial consequences when making an inventory decision. Second, for the traditional inventory problem, there exist very simple yet powerful models (e.g., Newsvendor, base-stock, etc.) that illustrate how inventory decision is affected by the system parameters. In the recent financial crisis, practitioners advocate the importance of interdisciplinary study between operations and accounting/finance. Yet, there are no simple models that help managers understand their relationships and manage the system. In this paper, we make a step to fill this void.

We formulate the inventory system with trade credit into a multi-state dynamic program that keeps track of inventory level, cash balance, as well as different ages of A/P and A/R within the payment and collection periods, respectively. In addition to standard inventory-related costs, cash-related costs are incurred in each period. The cash-related cost is composed of a default penalty cost due to a cash shortage and an interest gain (negative cost) due to excess cash after inventory payments. The objective is to obtain an inventory policy that maximizes the working capital at the end of the horizon. We first show that maximizing the end-of-horizon working capital is equivalent to minimizing the total inventory-related cost and cash-related cost in the horizon. We prove that the optimal policy is a state-dependent, order-up-to policy.

From a perspective of implementation, it would be difficult to execute a state-dependent policy. Thus, we approximate the model by simplifying the future cash state. With this simplification, we introduce the notation of working capital (= cash + inventory + accounts receivable - accounts payable) to reduce the problem to a two-state dynamic program. When the payment period is shorter than the collection period, we prove that a myopic policy is optimal for this simplified model when demand is non-decreasing. This myopic policy has a very simple structure. Let c be the unit purchase cost. The policy is operated under two control parameters (d, S), $d \leq S$: the firm reviews its working capital level and inventory position at the beginning of each period; if the working capital is lower (resp., higher) than the default threshold level cd (resp., the base-stock level cS), the firm places an order to bring its inventory position up to d (resp., S); if the working capital level is between cd and cS, the firm orders up to the working capital level (in inventory units). When the payment period is longer than the collection period, the firm's payment beyond the collection period depends on the future cash inflows, which in turn depend on the future random demands. This feature leads to a more generalized (d, S) policy, referred to as the $(\mathbf{d}, \mathbf{a}, S)$ policy. Depending on the length of the payment period and the collection period, we then apply either the

(d, S) policy or the $(\mathbf{d}, \mathbf{a}, S)$ policy as our heuristic. A numerical study suggests that the heuristic is very effective when comparing to the exact system. The heuristic policy resembles practical working capital management under which firms make inventory decisions according to the working capital level (Aberdeen Group 2009).

We summarize the main contributions. First, managers are hindered from integrating accounts payable/receivable into the inventory policy due to the typical organizational structure of the firm (i.e., the former is a function of a treasurer, and the latter an operations manager). These two functions need to be aligned when the firm has a financial shortage. We present a model that captures the dynamics between inventory decisions and accounts payable/receivable resulted from trade credit terms, and provide a simple and implementable heuristic policy. The policy suggests that, in addition to traditional inventory parameters, a firm should consider the entire working capital, the default penalty cost rate, and the interest return rate when making the inventory decision. This result augments the current scope of operations by incorporating accounting concepts. We also assess the value of considering financial flows when making the inventory decision in a numerical study.

Second, our model shows that maximizing a firm's end-of-horizon working capital is equivalent to minimizing the total inventory-related and cash-related cost in the horizon. When the default penalty is zero or the system working capital is high (i.e., cash is ample), the model reduces to the traditional inventory model and the heuristic policy degenerates into a base-stock policy. With this connection, we can provide a clear economic meaning of cost parameters for the traditional model. For example, the holding cost rate is composed of the physical holding cost and the cost of capital determined by the interest return rate. On the other hand, when the default penalty is sufficiently large, our policy suggests the firm order up to the working capital level, which is equivalent to the solution obtained from the cash-constrained model (Bendavid et al. 2012). We believe that the real-world practice falls between these two extremes, and our model reflects this generality. In addition, the heuristic policy parameters have closed-form, Newsvendor-type expressions, which facilitate classroom teaching and provide a clear intuition for the impact of system parameters on the inventory decision.

Third, our model reveals several insights that are consistent with empirical findings in the finance literature. For example, our policy indicates that a firm may choose to default on the payment if its current working capital level is low. This explains why payment defaults are commonly observed in practice as the default penalty cost is usually small (Cuñat and Garcia-Appendini 2012). In addition, we show that it is more beneficial for the firm to extend the payment period with its supplier if the firm also offers trade credit to its customer. This finding explains why there is a positive correlation between a firm's upstream and downstream credit periods (Fabbri and Klapper 2009).

Lastly, a firm often wishes to extend the payment period and shorten the collection period to reduce its cash conversion cycle (cash collection periods + on-hand inventory in periods - inventory payment periods). We can quantify this benefit through our model. However, extending the payment period may induce the supplier to increase the unit wholesale price, which may not benefit the buyer. Our model can generate an isocost curve for different pairs of purchase prices and payment periods, which provides guidance for firms to offer an early payment in exchange for a lower purchase price.

2 Literature Review

Our paper is related to a few research topics summarized below. In particular, in §2.4 we shall discuss several empirical findings, which set the stage for our model assumptions.

2.1 Inventory Models with Trade Credit

This literature is categorized based on whether a single- or multi-period problem is considered. For the single-period model, Zhou and Groenevelt (2008) consider the impact of financial collaboration in a third-party supply chain. They find that the total supply chain profit with bank financing is higher than that with open account (trade credit) financing. Kouvelis and Zhao (2012) consider a capital-constrained retailer which replenishes from a supplier. They show that a risk-neutral supplier should always finance the retailer at a rate less than or equal to the risk-free rate. Yang and Birge (2012, 2013) study how different priority rules of order repayment influence trade credit usage. They consider bankruptcy default, which is different from the payment (illiquidity) default assumed in our model.

For the multi-period model, this literature is further categorized based on how trade credit is modeled. One category is to characterize the impact of trade credit on the inventory holding cost rate. This literature implicitly assumes that cash is always available so cash dynamics are not explicitly modeled. Beranek (1967) uses a lot-size model to illustrate how a firm's inventory holding cost should be adjusted according to the firm's actual financial arrangements. Maddah et al. (2004) investigate the effect of permissible delay in payments on ordering policies in a periodic-review (s, S) inventory model with stochastic demand. They develop approximations for the policy parameters. Gupta and Wang (2009) consider a stochastic inventory system where the trade credit term is modeled as a non-decreasing holding cost rate according to an item's shelf age. Under the assumption that the full payment is made when the item is sold, they prove that a base-stock policy is optimal. Huh et al. (2011) and Federgruen and Wang (2010) generalize the results of Gupta and

Wang. Song et al. (2014) consider a retailer which replenishes from a supplier in a supply chain. Both firms implement a base-stock policy. They investigate how the holding cost rate is affected by the different payment and collection time epochs and the coordination issues.

Another category, which is more related to our model, is to explicitly characterize cash flow dynamics resulted from the trade credit terms. Haley and Higgins (1973) expand Beranek's model and consider the problem of jointly optimizing inventory decision and payment times when demand is deterministic and inventory is financed with trade credit. Schiff and Lieber (1974) consider the problem of optimizing inventory and trade credit policy for a firm where the demand is deterministic but depends on the credit term and inventory level. Bendavid et al. (2012) study a firm whose replenishment decisions are constrained by the working capital requirement. Their model is similar to ours in that they also consider how inventory replenishment is affected by the payment and the collection periods. However, their model considers i.i.d demand and implements a base-stock policy with inventory ordering subject to a hard constraint on the amount of working capital. Thus, no defaults are allowed. They characterize the dynamics of system variables and obtain the optimal base-stock level via a simulation approach. Modeling cash flows is necessary in these models because cash is a system state subject to financial constraints or penalties, which in turn affect operational decisions.

2.2 Interface of Operations and Finance

There has been an emerging research stream that aims to study problems that involve both operations and finance without considering trade credit. Xu and Birge (2004) analyze the interactions between a firm's production and financing decisions as a tradeoff between the tax benefits of debt and financial distress costs. Xu and Birge (2006) propose an integrated corporate planning model, which extends the forecasting-based discount dividend pricing method into an optimization-based valuation framework to make production and financial decisions simultaneously for a firm facing market uncertainty. Chao et al. (2008) consider a self-financed retailer who replenishes inventory in a finite horizon. Tanrisever et al. (2012) explore the tradeoff between investment in process development and reservation of cash in order to avoid bankruptcy for a start-up firm. They provide managerial insights by characterizing how to create operational hedges against the bankruptcy risk. Boyabatli et al. (2013) study the impact of budget constraints on the technology choice. They show that ignoring financial constraints may lead to mis-specifications. Li et al. (2013) study a dynamic model in which inventory and financial decisions are made simultaneously in order to maximize the firm's value – the expected present value of dividends minus total capital subscriptions. Luo and Shang (2015) integrate material and cash flows in a supply chain. They characterize the optimal joint inventory and investment policy and investigate the value of cash pooling. Chen et al. (2015) study the preservation of supermodularity properties for a class of two-dimensional parametric optimization problems. Their results can simplify the proofs in Chao et al. (2008).

2.3 Inventory Models with Capacity and Advanced Demand Information

Our model is related to two streams of inventory problems. The (inventory-equivalent) working capital level serves as an upper bound for the inventory order up-to level under the (d, S) policy. In this regard, our model is related to the capacitated inventory model. We refer the reader to Tayur (1997) for a review and Levi et al. (2008) for recent developments. The other stream is inventory systems with advance demand information. The incoming and outgoing cash flows in accounts receivable and payable pipelines can be viewed as advanced cash flow information. For this research stream, see Özer and Wei (2004) and references therein.

2.4 Finance Literature on Trade Credit

The motivation and several key assumptions of our model are based on the following empirical finance literature.

2.4.1 One-Part Trade Credit

Trade credit is widely used for business transactions in supply chains, and is the single most important source of external finance for firms (Petersen and Rajan 1997). It appears on the firm's balance sheet and accounts for about one half of the short-term debt in two samples of UK and US firms (Cuñat 2007). In the finance literature, there have been various theories, such as order incentives (Schwartz 1974), taxes (Brick and Fung 1984), transaction costs (Ferries 1981), and information asymmetries (Smith 1987, Lee and Stowe 1993) that explain the existence of trade credit; see Cuñat and Garcia-Appendini (2012) for an excellent review. Trade credit is one of the key sources of funding for small, entrepreneurial firms that lack collateral and credit history. It is well documented that trade credit is more common among newly created firms and those with less tangible assets (Berger and Udell 1998, Elliehausen and Wolken 1993). Among various types of trade credit contracts, the one-part (net term) is the simplest form. Cuñat (2007) indicates that there is a big portion of firms that use one-part contracts.[1] Our paper takes one-part trade credit as a premise and aims to investigate its impact on a firm's inventory policy and the resulting operating cost. Finally, two-level trade credit is commonly seen in practice, i.e., firms often provide trade credit to its customer while receiving one from its supplier. Fabbri and Klapper (2009) find evidence of a

[1] According to the 1998 NSSBF survey, 49% of the trade credit contracts are one-part.

positive correlation between a firm's upstream and downstream credit period length. Guedes and Mateus (2009) examine trade credit linkages on the propagation of liquidity shocks in supply chains. These papers motivate us to study two-level trade credits and our numerical results also echo their findings.

2.4.2 Late Payment and Default Penalty Cost

According to Table 1 in Cuñat and Garcia-Appendini (2012), late payments are not uncommon – In the category of net term 21-30 days,[2] 17.2% of the firms make payment after the due date. The same table also shows that the cost of late payment is minimal. Specifically, the average surcharge for each delayed dollar over the time after the due date is about 0.92 cents in the same category. In fact, many suppliers do not charge an explicit penalty for late payment. This distinctive feature shows how trade credit repayment contains an extra degree of flexibility, which can be extremely useful for small and entrepreneurial firms that have more volatile sales and are financially more fragile. One reason that the supplier does not charge for the payment default is the on-going relationships with the buyer (Cuñat 2007). Wilson and Summers (2002) provide rationale of sustaining this long-term business relationship. These findings support our model assumption that the supplier continues to do business with the firm which may occasionally default on the payment. Although the late payment causes little monetary penalty to the buyer, it is necessary to be factored in when making inventory decisions as frequent default behavior will hurt a firm's credit record, making bank finance or future transactions difficult in a later stage (Cook 1999; Garcia-Appendini 2007). Boissay and Gropp (2013) investigate liquidity shocks for small-sized French firms. They find that the payment default in a supply chain stops when it reaches firms that are large and have access to financial markets. In light of this, we introduce a default penalty cost to the model, which can be viewed as a "backorder" penalty cost charged on the unfilled payment.

2.4.3 Fixed Payment Period

Our model assumes that once the firm and its business partners have agreed upon a net term contract, the payment periods are fixed within the horizon. This is consistent with an empirical finding in Ng et al. (1999), where the trade credit terms (payment period in our context) may be different across industries, but they are relatively stable within each industry and along time. For example, the authors find that net 30 (i.e., pay in full within 30 days) is the most common net term contracts. Nevertheless, credit policy is an organizational design choice and firms have incentives to offer an early payment (say, pay-on-shipment) in exchange for a lower purchase price before agreeing

[2] Net 21-30 days represents the most common one-part trade credit used by the firms surveyed in the 1998 SSBF.

on the trade credit contract. Our model provides a tradeoff between the purchase price and the payment period, which can be used to assist in negotiation.

The rest of this paper is organized as follows. §3 describes the model and formulates the corresponding dynamic program. §4 focuses on the model with a longer collection period and derive the (d, S) policy. §5 considers the model with a longer payment period and presents the $(\mathbf{d}, \mathbf{a}, S)$ policy. §6 discusses the qualitative insights through numerical studies. §7 concludes. Appendix A shows the optimality of the $(\mathbf{d}, \mathbf{a}, S)$ policy. Appendix B provides all proofs. Throughout this paper, we define $x^+ = \max(x, 0)$, $x^- = -\min(x, 0)$, $a \vee b = \max(a, b)$, and $a \wedge b = \min(a, b)$.

3 The Model

We consider a finite-horizon, periodic-review inventory system where a firm orders from its supplier and sells to its customer. A one-part trade credit contract is employed for transactions with its upstream and downstream partners. That is, the firm pays its supplier after a payment period following the delivery of goods, and receives cash from its customer after a collection period following the demand. In accounting, the inventory payment period (sales collection period) is also referred to as the *payable (receivable) conversion period* or *days purchases (sales) outstanding.* The payment and collection periods jointly affect the *cash conversion cycle* (CCC), which is defined as

$$\text{CCC} = \text{Inventory conversion period} + \text{Collection period} - \text{Payment period}.$$

Transactions based on trade credit affect a firm's accounts payable (A/P) and accounts receivable (A/R). Table 1 lists the four events and the corresponding changes in inventory and cash levels, as well as accounts payable and receivable.

Transaction	**Inventory/Cash flow**	**Accounting Variables**
Receiving X units of inventory	Inventory ↑ X	A/P ↑ \$X
Selling Y units of inventory	Inventory ↓ Y	A/R ↑ \$Y
Paying \$X to the supplier	Cash ↓ \$X	A/P ↓ \$X
Collecting \$Y from the customer	Cash ↑ \$Y	A/R ↓ \$Y

Table 1: Events and accounting variables associated with CCC

We now formalize the above description into our model. Since the focus is on cash and inventory dynamics under trade credit, for simplicity and without loss of generality, we assume that the shipping lead time is zero.[3] Let m be the payment period and n be the collection period. We count the time forward, i.e., $t = 0, 1, 2$, etc. The sequence of events is as follows: At the beginning of

[3]The analysis can be extended to the general lead time case by assuming x as the inventory position before ordering. This will shift the inventory-related cost to a later period but will not affect the policy and the results.

period t, (1) inventory order decision is made and a new A/P is generated; (2) shipment arrives; (3) payment due in this period (corresponding to the inventory ordered in period $t-m$) is made to the supplier; (4) a default penalty cost is incurred in case of insufficient payment or an interest return is gained in case of a positive cash level; demand is realized during the period and a new A/R is generated. Customer payment due in this period (corresponding to the sales in period $t-n$) is collected; at the end of the period, all inventory related costs are calculated. The objective is to maximize the firm's working capital at the end of the T-period horizon.

Customer demand in period t is modeled as a nonnegative random variable D_t with probability density function (p.d.f.) f_t, cumulative distribution function (c.d.f.) F_t, mean μ_t and variance σ_t^2. The demand is stochastic and independent between periods. We assume that the unsatisfied demand is fully backlogged.

Define the state and decision variables at the beginning of period t:

$$
\begin{aligned}
x_t &= \text{net inventory level before Event (1);} \\
y_t &= \text{inventory position after Event (2);} \\
w'_t &= \text{net cash level before Event (3);} \\
\mathbf{P}_t &= (P_{t-m}, ..., P_{t-1})\text{: } m\text{-dimensional vector of accounts payable;} \\
\mathbf{R}_t &= (R_{t-n}, ..., R_{t-1})\text{: } n\text{-dimensional vector of accounts receivable.}
\end{aligned}
$$

Here, P_{t-i} and R_{t-j} denote the A/P and A/R created in period $t-i$ and $t-j$, respectively, for $i = 0, 1, ..., m$ and $j = 0, 1, ..., n$. So P_{t-m} and R_{t-n} are the most aged A/P and A/R, while P_t and R_t are A/P and A/R created in the current period. Figure 1 shows these system variables in period t with the material and cash flows in solid and dashed arrows, respectively.

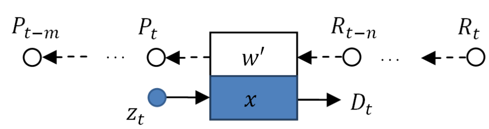


Figure 1: The base model with material flow and cash flow

Let p be the unit sales price, c be the unit purchase cost, h be the physical holding cost rate, r be the interest return rate for the positive net cash level after payment, and e be the default penalty cost rate for the negative cash level after payment. To avoid the firm to gain interest returns by intentionally defaulting the payment, we assume that $e > r$. In each period t, the accounts payable created due to the inventory order quantity is $P_t = c(y_t - x_t)$, $y_t \geq x_t$. The accounts receivable generated from the random sales is $R_t = p\min\{D_t, y_t\}$. The interest return on the net cash after payment is $r(w_t^{'} - P_{t-m})^+$, and the default penalty cost is $e(P_{t-m} - w_t^{'})^+$. Define the working capital w_t at the beginning of each period t as

$$w_t = cx_t + w_t^{'} - \sum_{i=1}^{m} P_{t-i} + \sum_{j=1}^{n} R_{t-j}. \tag{1}$$

Then, the evolution of the system variables are

$$\begin{aligned}
x_{t+1} &= y_t - D_t, &(2)\\
w_{t+1}^{'} &= w_t^{'} + R_{t-n} - P_{t-m} - h(y_t - D_t)^+ - e(P_{t-m} - w_t^{'})^+ + r(P_{t-m} - w_t^{'})^-, &(3)\\
\mathbf{P}_{t+1} &= (\mathbf{P}_t^{-1}, c(y_t - x_t)), &(4)\\
\mathbf{R}_{t+1} &= (\mathbf{R}_t^{-1}, p\min\{y_t, D_t\}), &(5)
\end{aligned}$$

where we denote $\mathbf{P}_t^{-1}$ as vector $\mathbf{P}_t$ without the first element (same for $\mathbf{R}_t^{-1}$). Also define

$$g_t(y_t) = h(y_t - D_t)^+ + p(y_t - D_t)^-, \tag{6}$$

$$\nu_t(P_{t-m}, w_t^{'}) = e(P_{t-m} - w_t^{'})^+ - r(P_{t-m} - w_t^{'})^-. \tag{7}$$

From Equation (1) to (7), after some algebra, the working capital in period $t+1$ can be expressed as

$$w_{t+1} = w_t + (p-c)D_t - g_t(y_t) - \nu_t(P_{t-m}, w_t^{'}), \tag{8}$$

We refer to $g_t(y_t)$ as the inventory-related cost and $\nu_t(w_t^{'}, P_{t-m})$ as the cash-related cost. Equation (8) states that the working capital in period $t+1$ is equal to the working capital in period t plus net cash flows. With this result, it can be shown that the end-of-horizon working capital level is equal to the initial working capital plus the total net cash flows within the horizon. More specially,

$$w_{T+1} = w_1 + \sum_{t=1}^{T}(p-c)D_t - \sum_{t=1}^{T} g_t(y_t) - \sum_{t=1}^{T} \nu_t(P_{t-m}, w_t^{'}).$$

Note that $\mathsf{E}[(p-c)D_t]$ is a constant and w_1 is the initial working capital. Thus, maximizing the expected end-of-horizon working capital, $\mathsf{E}[w_{T+1}]$, is equivalent to minimizing the expected total inventory related cost and cash related cost within the horizon. That is,

$$\min_{y_t \geq x_t, \forall t} \quad \left\{ \mathsf{E}\left[\textstyle\sum_{t=1}^{T} \left(g_t(y_t) + \nu_t(P_{t-m}, w_t^{'}) \right) \right] \right\}. \tag{9}$$

The problem (9) can be solved from the dynamic program below. Denote $\hat{V}_t(x_t, w_t', \mathbf{P}_t, \mathbf{R}_t)$ as the minimum expected cost from period t to $T+1$ among all feasible policies.

$$\hat{V}_t(x_t, w'_t, \mathbf{P}_t, \mathbf{R}_t) = \min_{y_t \geq x_t} \left\{ \left(G_t(y_t) + \nu_t(P_{t-m}, w'_t) \right) + \mathsf{E}\hat{V}_{t+1}(x_{t+1}, w'_{t+1}, \mathbf{P}_{t+1}, \mathbf{R}_{t+1}) \right\}, \tag{10}$$

where $G_t(y_t) = \mathsf{E}[g_t(y_t)]$, and w'_{t+1} follows the dynamics in (3). The terminal condition is

$$\hat{V}_{T+1}(x_{T+1}, w'_{T+1}, \mathbf{P}_{T+1}, \mathbf{R}_{T+1}) = 0.$$

One can view that our model is a generalization of the classic inventory model in which there are no trade credit contracts and cash constraints. More specifically, consider the special case where $m = n = 0$ and $c(y_t - x_t) \leq w'_t$ in each period. In this case, Equation (8) becomes

$$w_{t+1} = (1+r)w_t + (p-c)D_t - rcy_t - g_t(y_t),$$

and

$$\mathsf{E}[w_{T+1}] = \mathsf{E}\left[(1+r)^T w_1 + \sum_{t=1}^{T} (1+r)^{T-t} \left(rc \cdot y_t + g_t(y_t) \right) \right].$$

Thus, the problem (9) is reduced to the following problem:

$$\begin{aligned} &\min_{y_t \geq x_t, \forall t} \mathsf{E}\left[\sum_{t=1}^{T} (1+r)^{T-t} \left(rc \cdot y_t + g_t(y_t) \right) \right] \\ =& \min_{y_t \geq x_t, \forall t} (1+r)^{T-1} \left(\mathsf{E}\left[\sum_{t=1}^{T} \alpha^{t-1} \left(rc \cdot y_t + g_t(y_t) \right) \right] \right), \end{aligned}$$

where $\alpha = 1/(1+r)$ the discount rate. The optimal order quantity can be obtained from the following equivalent dynamic program: $f_{T+1}(\cdot) = 0$, and

$$f_t(x) = \min_{y \geq x} \left\{ \mathsf{E}[(h+rc)(y-D_t)^+ + (p-rc)(y-D_t)^- + \alpha f_{t+1}(y-D_t)] \right\}. \tag{11}$$

This formulation provides a clear economic explanation for the cost parameters in the classic inventory model: The first term on the right-hand side of (11) represents the expected holding cost. The holding cost rate is the sum of the physical inventory holding cost and the loss of the interest return by investing the inventory. The second term is the average backorder cost. The backorder cost rate is the product selling price minus the interest return due to inventory purchase. Notice that the single-period cost function in (11) does not include the purchase cost $c(y_t - x_t)$. This is because the total working capital does not change after inventory purchase – the increased inventory value in the working capital is equal to the decreased cash amount due to inventory purchase. It is easy to show that a myopic solution s_t^* is optimal to (11), provided that the demand is non-decreasing in t and $x_1 \leq y_1$. Here, s_t^* is the solution of the following equation:

$$P(D_t \leq s) = F(s) = \frac{p - rc}{p + h}.$$

This solution is exactly the same as that obtained from the traditional inventory model by assuming that a free return on excess inventory or a unit purchase cost incurred for each backlogged unit at the end of the horizon, i.e., the terminal value is $-cx_{T+1}$ (e.g., Porteus (2002), p.70).[4] Our formulation does not require such an assumption on the terminal condition.

The model formulated in (10) has a state space of $m+n+2$ dimensions. The proposition below shows that the optimal value function $\hat{V}_t$ is jointly convex. Let $s_t(x_t, w_t', \mathbf{P}_t, \mathbf{R}_t)$ denote the optimal solution.

Proposition 1. *The optimal policy for the inventory model in (10) is a state-dependent, order-up-to policy, where the target inventory stocking level is* $s_t(x_t, w_t', \mathbf{P}_t, \mathbf{R}_t)$.

A state-dependent policy is difficult to implement as computing the optimal solution requires a significant computational effort due to curse of dimensionality. Also, it reveals little insight of managing the system. In the subsequent sections, we aim to resolve this issue. Our idea is to present a simplified model that eliminates the curse of dimensionality issue and show that a simple policy is optimal to this simplified model. This optimal policy will then be used as the heuristic policy for the original system.

3.1 Simplified Model

For the problem in (9), the order decision y_t affects the inventory holding and backorder cost $g_t(y_t)$ in period t and the cash-related cost $\nu_{t+m}(w_{t+m}^{'}, P_t)$ in period $t+m$. Although this problem has a similar structure as the classic inventory model with lead time L, in which the order decision in period t affects the ordering cost in period t and the inventory-related cost in period $t+L$, the cash level $w_{t+m}^{'}$ is jointly determined by the future inventory decisions y_{t+m-i}, $i = 1, ..., m$. More specifically, applying Equation (3) recursively, we have

$$w_{t+m}^{'} = w_t^{'} + \sum_{i=1}^{m} R_{t+m-n-i} - \sum_{i=1}^{m} P_{t-i} - \sum_{i=1}^{m} h(y_{t+m-i} - D_{t+m-i})^+ + \sum_{i=1}^{m} \nu_{t+m-i}(P_{t-i}, w_{t+m-i}^{'}).$$

As shown, $w_{t+m}^{'}$ is jointly determined by y_{t+m-i} through $R_{t+m-n-i} = p\min\{y_{t+m-n-i}, D_{t+m-n-i}\}$, the inventory holding cost, and the cash-related cost. This dependence leads to curse of dimensionality.

[4]The interest gain in our model occurs at the beginning of each period after inventory payment. This sequence of event corresponds to a traditional model with lead time of one period and the discount rate $\alpha = 1/(1+r)$. See Proposition 3 of Shang (2012) for the myopic solution with a positive lead time.

To resolve this issue, we simplify the system by making the following two assumptions.

Assumption 1. *The inventory holding and cash-related costs are omitted in the cash dynamics.*

Assumption 2. *The revenue (or accounts receivable) received in period t is pD_t.*

The inventory holding cost and interest returns are relatively small compared to the revenue so omitting them in the cash dynamics may not affect the system behaviors. Assumption 2 is a common assumption in the revenue management literature. It assumes that the customer pays the order in full as the backlogged units will eventually be filled in the backorder model. With these two assumptions in place, the cash dynamic in (3) and the accounts receivable in (5) become

$$w'_{t+1} = w'_t + R_{t-n} - P_{t-m},$$
$$\mathbf{R}_{t+1} = (\mathbf{R}_t^{-1}, pD_t).$$

We use the above two state transitions to solve the problem in (10). We refer to the resulting system as the *simplified model*. Depending on the order of m and n, we can transform the problem in (10) into two different dynamic programs, which will be discussed in §4 and §5, respectively.

4 The System with Longer Collection Period

For the system with a longer collection period, i.e., $m \leq n$, the inventory decision y_t affects the cash-related cost in period $t+m$. More specifically, under Assumptions 1 and 2,

$$\begin{aligned} w'_{t+m} &= w'_t + \sum_{i=1}^{m} R_{t+m-n-i} - \sum_{i=1}^{m} P_{t-i} = \left(w'_t + \sum_{j=1}^{n} R_{t-j} - \sum_{i=1}^{m} P_{t-i} \right) - \sum_{k=1}^{n-m} R_{t-k} \\ &= w_t - \sum_{k=1}^{n-m} R_{t-k}. \end{aligned} \tag{12}$$

The cash-related cost in period $t+m$ is determined by $(P_t - w'_{t+m})$, which is

$$P_t - w'_{t+m} = cy_t - \left(w_t - \sum_{k=1}^{n-m} R_{t-k} \right). \tag{13}$$

Notice that $\sum_{k=1}^{n-m} R_{t-k}$ is known in period t. Thus, we define $\underline{w}_t = w_t - \sum_{k=1}^{n-m} R_{t-k}$ as the working capital in period t excluding the known accounts receivable in periods $t-n+m$, ..., $t-1$.

The optimal solution to the approximate system can be obtained from the following dynamic program: $V_{T+1}(x, \underline{w}) = 0$, and

$$V_t(x, \underline{w}) = \min_{y \geq x} \Big\{ G_t(y) + \nu_t(cy, \underline{w}) + \mathsf{E}[V_{t+1}(y - D_t, \underline{w} + R_{t-n+m} - cD_t)] \Big\}. \tag{14}$$

For notational simplicity, we do not explicitly include the known accounts receivables $R_{t-n+m}, \ldots, R_{t-1}$ in the state space. For the special case of $m = n$, the working capital dynamics become $\underline{w}_{t+1} = \underline{w}_t + (p-c)D_t$. Without confusion, we omit the time index t in the state variables in the sequel. We can show that the optimal solution for the above dynamic program is a state-dependent policy.

Proposition 2. *(1) $V_t(x, \underline{w})$ is jointly convex in x and $\underline{w}$; (2) A state-dependent base stock policy $s_t(x, \underline{w})$ is optimal, i.e., order up to $s_t(x, \underline{w})$ if $x \le s_t(x, \underline{w})$ and do not order otherwise.*

Proposition 2 shows that by introducing the concept of working capital and employing the two assumptions, we can reduce the original problem from $(m+n+2)$ states to two states. This makes the computation possible. However, from a perspective of implementation and revealing insights, a state-dependent policy is not ideal. Below we shall introduce a simple and implementable policy.

4.1 Single-Period Problem

We first solve the single-period problem in (14) and obtain the corresponding myopic policy, referred to as the (d, S) policy. We then show the (d, S) policy is indeed optimal for the finite horizon problem in (14) when the demand is stochastically non-decreasing. As we shall see in §6.1, the policy remains very effective for the nonstationary demand case in our numerical study.

Without considering the constraint, the single-period minimization problem in (14) can be written as

$$v_t(\underline{w}) = \min_y \left\{ G_t(y) + e(cy - \underline{w})^+ - r(cy - \underline{w})^- \right\}, \tag{15}$$

where $G_t(y_t) = \mathsf{E}[g_t(y_t)]$. To facilitate our discussion, we first define the control parameters:

$$d_t = \left\{ y : \frac{\partial}{\partial y} G_t(y) = -ec \right\}; \qquad S_t = \left\{ y : \frac{\partial}{\partial y} G_t(y) = -rc \right\}. \tag{16}$$

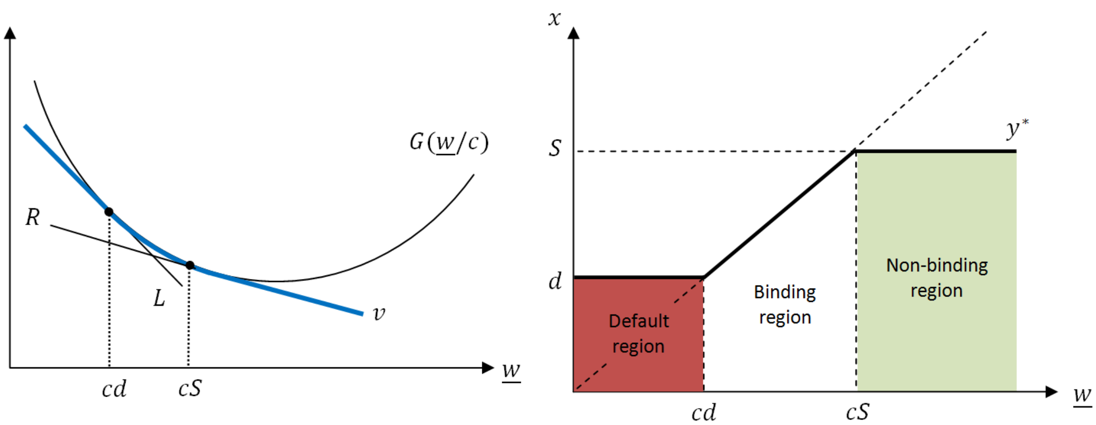
Or equivalently,

$$F_t(d_t) = \frac{p - ec}{p + h}; \qquad F_t(S_t) = \frac{p - rc}{p + h}. \tag{17}$$

To solve the problem in (15), we consider three cases; see Figure 2(a).

Case 1. When $\underline{w} \le cd_t$, the system's working capital is lower than the default threshold cd_t. In this case, the firm has an incentive to order up to d_t as the marginal backorder cost and interest return outweighs the marginal holding and default penalty cost. Thus, we have $v_t(\underline{w}) = L_t(\underline{w}) = G_t(d_t) - e(\underline{w} - cd_t)$.

Case 2. When $cd_t < \underline{w} \le cS_t$, the system is constrained by the working capital. It is optimal to order up to $\underline{w}/c$ as ordering either less or more will lead to a higher cost than $G_t(\underline{w}/c)$. Thus,



(a) The myopic problem and the optimal control parameters

(b) The optimal (d, S) policy

Figure 2: The optimal solution of the transformed λ-model

$v_t(\underline{w}) = G_t(\underline{w}/c)$.

Case 3. When $cS_t < \underline{w}$, the system has ample working capital and orders up to the target base-stock S_t. In this case, there is extra cash left after ordering, which yields an interest return of $r(\underline{w} - cS_t)^+$. In this case, $v_t(\underline{w}) = R_t(\underline{w}) = G_t(S_t) - r(\underline{w} - cS_t)$.

As a result, Equation (15) becomes

$$v_t(\underline{w}) = \left\{ \begin{array}{ll} L_t(\underline{w}), & \text{if } \underline{w} \leq cd_t \\ G_t(\underline{w}/c), & \text{if } cd_t < \underline{w} \leq cS_t \\ R_t(\underline{w}), & \text{if } cS_t < \underline{w} \end{array} \right\}, \tag{18}$$

Denote y_t^* as the resulting optimal order-up-to level under the (d, S) policy, that is,

$$y_t^*(\underline{w}) = (d_t \vee \underline{w}/c) \wedge S_t. \tag{19}$$

We define the region where the initial inventory level x is less than or equal to $y_t^*(\underline{w})$ as follows:

$$\mathcal{B}_t = \left\{(x, \underline{w}) \in \Re^2 \mid x \leq y_t^*(\underline{w})\right\}. \tag{20}$$

Figure 2(b) depicts the piecewise linear function $y_t^*(\underline{w})$. By definition, the band $\mathcal{B}$ covers the area below $y_t^*(\underline{w})$ on the x-$\underline{w}$ plain. If $x \leq y_t^*(\underline{w})$, the (d, S) policy is optimal for the myopic problem. We summarize this result below.

Proposition 3. *The (d, S) policy is optimal for the myopic problem in (15). The firm monitors its inventory level x and working capital $\underline{w}$ at the beginning of each period. If $\underline{w}/c \leq d_t$, the firm orders up to d_t; if $d_t < \underline{w}/c \leq S_t$, the firm uses up all cash and orders inventory up to $\underline{w}/c$; if $\underline{w}/c > S_t$, the firm orders inventory up to S_t.*

4.2 Finite-Horizon Problem

We next show that the (d, S) policy is indeed optimal for the finite-horizon problem when $n \geq m$.

Proposition 4. *If D_t is stochastically increasing in t,*

(a) the control parameters d_t and S_t are non-decreasing in t and $d_t \leq S_t$ for all t;

(b) $V_t(x, \underline{w}) = W_t(\underline{w})$ for all t and $(x, \underline{w}) \in \mathcal{B}_t$, where

$$W_t(\underline{w}) = v_t(\underline{w}) + \mathsf{E}W_{t+1}(\underline{w} + R_{t-n+m} - cD_t),$$

and $W_{T+1}(\underline{w}) = 0$; $W_t(\underline{w})$ is convex in $\underline{w}$;

(c) the (d, S) policy is optimal for the dynamic program in (14).

Proposition 4(a) implies that if the initial state $(x, \underline{w})$ falls in the band, the system states at the beginning of each period will remain in the band under the (d, S) policy. This property ensures the optimality of the myopic policy, which is similar to the classic inventory model.

The (d, S) policy reveals interesting insights on managing working capital. The firm will have a chance to default when $d_t > -\infty$. From (17), this scenario happens when $p \geq ec$. This explains why payment defaults are prevalent in practice as the penalty cost is often very small (see §2.4.2). In general, when the default penalty is sufficiently large, the firm will be less likely to default, so the resulting model is similar to the cash-constrained model (i.e., cash becomes a hard constraint that restricts the inventory decision) studied by Bendavid et al. (2012). On the other hand, when there is ample cash supply, the working capital can always be set equal to the ideal level S_t in (17), which is equal to s_t^*, the optimal base-stock level for the classical inventory model. In this case, the (d, S) policy is degenerated into the classic base-stock policy.

To formally characterize the firm's order strategy under default risk, we define the *default quantity* as $u^*(\underline{w}) = (cy^*(\underline{w}) - \underline{w})^+$. Figure 2(b) implies that $u^*(\underline{w})$ is decreasing in $\underline{w}$, meaning that the firm will default less if there is more working capital. This behavior echoes the empirical findings that the operational decisions of smaller firms are more aggressive and thus induce higher default risks.

5 The Model with Longer Payment Period

For the system with a longer payment period, i.e., $m > n$, we can derive similar cash and working capital dynamics as those in (12) and (13). More specifically,

$$w'_{t+m} = w'_t + \sum_{j=1}^{n} R_{t-j} - \sum_{i=1}^{m} P_{t-i} + \sum_{k=1}^{m-n} R_{t+k-1},$$
$$P_t - w'_{t+m} = cy_t - \left(w_t + \sum_{k=1}^{m-n} R_{t+k-1} \right).$$

The resulting dynamic program is $V_{T+1}(x, w) = 0$, and

$$\begin{aligned} V_t(x, w) = &\min_{y \geq x} \Big\{ G_t(y) + \mathsf{E}\Big[e\Big(cy - \Big(w + \sum_{k=1}^{m-n} R_{t+k-1} \Big) \Big)^+ - r\Big(cy - \Big(w + \sum_{k=1}^{m-n} R_{t+k-1} \Big) \Big)^- \\ &+ V_{t+1}(y - D_t, w + (p - c)D_t) \Big] \Big\}. \end{aligned} \tag{21}$$

Note that $\sum_{k=1}^{m-n} R_{t+k-1} = \sum_{k=1}^{m-n} pD_{t+k-1}$ is a random variable, which is unknown in period t. For ease of exposition, define $m' = m - n$ and $D_t^{m'} = \sum_{k=1}^{m'} D_{t+k-1}$. In addition, let $F^{m'}$, $f^{m'}$, $\mu^{m'}$, and $(\sigma^{m'})^2$ be the c.d.f., the p.d.f., mean, and variance of the random variable of $D^{m'}$, respectively. Moreover, denote $\bar{F}^{m'}$ and $\hat{F}^{m'}$ as the complementary cumulative distribution function (c.c.d.f.) and the loss function of random variable $D^{m'}$. That is, $\hat{F}^m(x) = \int_x^\infty \bar{F}^m(y)dy$. Equation (21) becomes

$$\begin{aligned} V_t(x, w) = &\min_{y \geq x} \Big\{ G_t(y) + \mathsf{E}\Big[e(cy - (w + pD_t^{m'}))^+ - r(cy - (w + pD_t^{m'}))^- \Big] \\ &+ \mathsf{E}V_{t+1}(y - D_t, w + (p - c)D_t) \Big\}. \end{aligned} \tag{22}$$

Similar to the model with a longer collection period, we can prove that a state-dependent order-up-to policy is optimal.

Proposition 5. *(1) $V_t(x, w)$ is jointly convex in x and w. (2) Let $s_t(x, w)$ be the optimal solution. The optimal policy is to order up to $s_t(x, w)$ if $x \leq s_t(x, w)$ and do not order otherwise.*

Again, one can compute the optimal policy by introducing the notion of working capital for this simplified model. With the same intention of making the system more transparent and manageable, we provide two simple policies below.

5.1 Linear Approximation

Although the problem in (22) has the same structure as that of (14), the term $pD_t^{m'}$ makes the expected cash-related cost

$$\mathsf{E}[\nu_t(cy, w + pD_t^{m'})] = \mathsf{E}[e(cy - (w + pD_t^{m'}))^+ - r(cy - (w + pD_t^{m'}))^-] \tag{23}$$

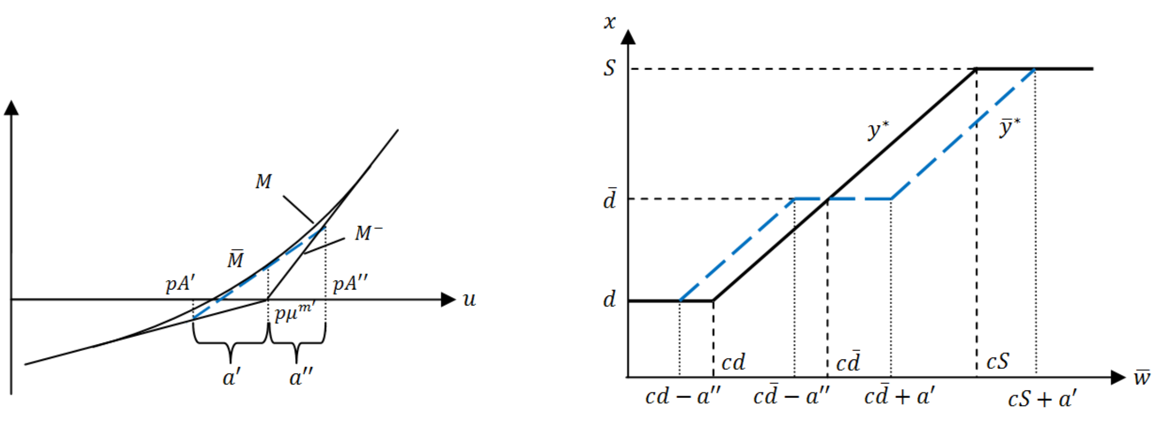
a general convex function (instead of a two-piece linear function in the model with a longer collection period). Thus, it is not possible to develop a simple policy for the model. To tackle this issue, we propose two types of piece-wise linear approximation on the cash-related cost function. To simplify the expression, define $u = cy - w$, and the expected cash-related cost can be rewritten as $M_t(u) = \mathsf{E}[e(u - pD_t^{m'}))^+ - r(u - pD_t^{m'}))^-]$. In the sequel, we suppress the time subscript without confusion.

Two-piece linear approximation

The first piece-wise linear approximation is generated by replacing the random variable $D^{m'}$ with the mean value $\mu^{m'}$ in the M function. More specifically,

$$M^-(u) = e(u - p\mu^{m'})^+ - r(u - p\mu^{m'})^-.$$

From Jensen's inequality, it is clear that $M^-(u) \leq M(u)$ for all u. Moreover, both functions have the same asymptotic slope e and r; see Figure 3(a).



(a) Piecewise linear approximations (b) The optimal (d, S) and $(\mathbf{d}, \mathbf{a}, S)$ policies

Figure 3: Linear approximations and optimal control policies

Three-piece linear approximation

The above two-piece linear approximation only characterizes the first moment of random variable

$D^{m'}$. Here, we further develop an approximation based on a three-piece linear function. This approximation, while more complicated to generate, takes into account the demand variability.

The construction of the three-piece linear function starts with a linear function $\Gamma(y)$, a tangent line to the convex curve M at point $(p\mu^{m'}, M(p\mu^{m'}))$. Denote the slope of Γ be $\bar{p}$. Then, Γ will intersect M^- with two points $(pA^{'}, M^-(pA^{'}))$ and $(pA^{''}, M^-(pA^{''}))$. We use the following three-piece linear function $\bar{M}(u)$ to approximate $M(u)$, where

$$\bar{M}(u) = \max\left\{M^-(u), \Gamma(u)\right\} = \left\{ \begin{array}{ll} M^-(u), & \text{if } u \le pA' \\ \Gamma(u), & \text{if } pA' < u \le pA'' \\ M^-(u), & \text{if } u > pA'' \end{array} \right\}. \tag{24}$$

Define the distance between $pA^{'}$ and $p\mu^{m'}$ as $a^{'}$ and that between $p\mu^{m'}$ and $pA^{''}$ as $a^{''}$. Proposition 6 shows that the variability of $D^{m'}$ is reflected in the distance, i.e., if the demand is more variable, the $\bar{M}$ function will be flatter, resulting in larger $a^{'}$ and $a^{''}$. The relationship between demand variability and $a^{'}$ or $a^{''}$ is shown in the following proposition.

Proposition 6. *The distances $a' = p\hat{F}^{m'}(\mu^{m'})/F^{m'}(\mu^{m'})$, and $a'' = p\hat{F}^{m'}(\mu^{m'})/\bar{F}^{m'}(\mu^{m'})$, where the loss function $\hat{F}^{m'}(\mu^{m'}) = (\sigma^{m'})^2 f(\mu^{m'})$ for most unimodal distribution functions.*[5]

We refer to the resulting model with M_t^- and $\bar{M}_t$ in place in (22) as the two-piece and three-piece approximate model, respectively.[6] We shall show the optimal policy for these approximate models when the demand is non-decreasing. The optimal policy will be used as the heuristic for the model with a longer payment period.

5.2 Heuristic Solutions

Define the *expected working capital* as $\bar{w} = w + p\mu^{m'}$, where the second term is the expected A/R within m' periods. We first derive the optimal solution to the two-piece approximate model. By replacing M_t with M_t^-, the resulting problem shares the same structure as the model with a longer collection period. Therefore, the (d, S) policy is optimal for the two-piece approximate model. The optimal policy is operated exactly the same as the (d, S) policy introduced in §4.1 except that the system monitors $\bar{w}$ instead of $\underline{w}$. The solid line in Figure 3(b) depicts this optimal base-stock policy:

$$y^*(\bar{w}) = (d \vee (\bar{w}/c)) \wedge S. \tag{25}$$

We next derive the optimal policy for the three-piece approximate model. By replacing M_t with

[5]The demand functions include, for example, Poisson, Geometric, Negative-Binomial, Exponential, Gamma, and Normal distributions.

[6]Clearly, the functions of M_t^- and $\bar{M}_t$ are lower bounds to the expected single-period cost function in (23). Thus, the resulting optimal cost is a lower bound to the optimal cost of the simplified model in (22).

$\bar{M}_t$ and w with $\bar{w}$, the dynamic program in (22) becomes

$$\begin{aligned}\bar{V}_t(x, \bar{w}) = &\min_{y \geq x} \Big\{ G_t(y) + \bar{M}_t(cy - \bar{w} + p\mu^{m'}) \\ &+ \mathsf{E}\bar{V}_{t+1}(y - D_t, \bar{w} + (p-c)D_t + p\mu_{t+m'} - p\mu_t) \Big\}. \end{aligned} \tag{26}$$

We first introduce the myopic policy, referred to as the $(\mathbf{d}, \mathbf{a}, S)$ policy, which consists of five control parameters $(\mathbf{d}, \mathbf{a}, S)$, where $\mathbf{d} = (d, \bar{d})$ and $\mathbf{a} = (a', a'')$. The firm implements a base-stock policy with the optimal base-stock level dependent on the expected working capital $\bar{w}$. More specifically, let $\bar{y}^*(\bar{w})$ be the optimal order-up-to level. Then,

$$\bar{y}^*(\bar{w}) = \left\{ \begin{array}{ll} d, & \text{if } \bar{w} \leq cd - a'' \\ (\bar{w} + a'')/c & \text{if } cd - a'' < \bar{w} \leq c\bar{d} - a'' \\ \bar{d}, & \text{if } c\bar{d} - a'' < \bar{w} \leq c\bar{d} + a' \\ (\bar{w} - a')/c, & \text{if } c\bar{d} + a' < \bar{w} \leq cS + a' \\ S, & \text{if } cS + a' < \bar{w} \end{array} \right\}. \tag{27}$$

We next illustrate how these control parameters are obtained. For fixed $\bar{w}$, the unconstrained single-period minimization problem in (26) can be written as

$$\bar{v}_t(\bar{w}) = \min_y \left\{ G_t(y) + \bar{M}_t(cy - \bar{w} + p\mu^{m'}) \right\}. \tag{28}$$

The control parameters a' and a'' are derived in Proposition 6. The base-stock S_t and default threshold d_t can be derived from (16). We refer to $\bar{d}_t$ as the *expected default threshold*, which can be obtained from

$$\bar{d}_t = \left\{ y : \frac{\partial}{\partial y} G_t(y) = -\bar{p}_t \right\}, \tag{29}$$

where $\bar{p}_t$ is the slope of Γ_t. The critical ratio of $\bar{d}_t$ can be expressed in a closed form.

Proposition 7. *The expected default threshold $\bar{d}_t$ satisfies*

$$F_t(\bar{d}_t) = \frac{p - (e-r)cF_t^{m'}(\mu_t^{m'}) - rc}{p + h}. \tag{30}$$

From the above expression and (16), we have $d_t \leq \bar{d}_t \leq S_t$, i.e., the default threshold d_t is lower than the expected default threshold $\bar{d}_t$ as the latter takes into account the expected A/R. Equation (29) also shows that the $(\mathbf{d}, \mathbf{a}, S)$ policy is a generalization of the (d, S) policy.

Proposition 8. *The $(\mathbf{d}, \mathbf{a}, S)$ policy in (27) is optimal for the myopic problem in (28).*

The dashed line in Figure 3(b) depicts the optimal base-stock $\bar{y}^*$ of the $(\mathbf{d}, \mathbf{a}, S)$ policy. In particular, if the firm's expected working capital $\bar{w}$ is lower than the expected default threshold, the firm will order more than its expected working capital, resulting in payment defaults in the future. For this reason, we term the region below $\bar{d}$ as the expected default region. On the other hand, if $\bar{w} > \bar{d}$, the firm will order less than its expected working capital level and hold extra cash on expectation. Notice that the over-order (under-order) deviation amount depends on a'' (a'), which is proportional to the variance of the aggregated demand. Denote the *expected optimal default quantity* as $\bar{u}^*(\bar{w}) = (c\bar{y}^*(\bar{w}) - \bar{w})^+$. Clearly, $\bar{u}^*(\bar{w})$ is decreasing with $\bar{w}$, implying that lower (higher) working level leads to more aggressive (conservative) inventory ordering decisions. This is consistent with the (d, S) policy.

In Appendix A, we show that under very general conditions, the $(\mathbf{d}, \mathbf{a}, S)$ policy is optimal for the three-piece approximate model in the finite-horizon problem when demand is stochastically non-decreasing.

The (d, S) and $(\mathbf{d}, \mathbf{a}, S)$ policy can serve as a heuristic for the model with a longer payment period. It is conceivable that the $(\mathbf{d}, \mathbf{a}, S)$ policy works better than the (d, S) policy, although the latter involves less control parameters, and thus is easier to implement. The performance gap between these two heuristic policies gets bigger when aggregated demand is more volatile. In practice, the (d, S) policy could serve as a simple alternative if the demand is less variable.

6 Numerical Study

In §6.1, we shall develop a lower bound to the optimal cost of the exact system. The purpose of developing this cost lower bound is to examine the effectiveness of the (d, S) policy and the $(\mathbf{d}, \mathbf{a}, S)$ policy.[7] In §6.2, we compare the (d, S) policy with two known inventory control policies in the literature. We illustrate the importance of collaboration between operations and accounting/finance departments when making the inventory decision. In §6.3, we discuss the impact of payment periods on the system's total cost.

6.1 Effectiveness of the Heuristic Policies

Nondecreasing Demand Case

We propose the (d, S) policy as a heuristic for the system with a longer collection period and the $(\mathbf{d}, \mathbf{a}, S)$ policy for the system with a longer payment period. To examine the effectiveness of the proposed heuristic, we develop a lower bound to the optimal cost of the *exact* system defined in (10) with the nondecreasing demand. Below we only sketch the idea. A detailed proof is available from the authors.

[7] Due to curse of dimensionality, it is computationally infeasible to compare the heuristic with the exact system.

We rewrite (3) as

$$\begin{aligned} w'_{t+1} &= (1+r)(w'_t - P_{t-m}) + R_{t-n} - h(y_t - D_t)^+ - (e-r)(P_{t-m} - w'_t)^+ \\ &\leq (1+r)w'_t - P_{t-m} + R_{t-n}. \end{aligned}$$

Note also that $R_t = p\min\{y_t, D_t\} \leq pD_t$. It can be shown that $\hat{V}_{t+1}$ decreases with w'_{t+1} and R_t, i.e., the firm incurs less cost if it had more on-hand cash or A/R. Simultaneously replacing (3) and (5) with the following two equations will result in a cost lower bound of the original system.

$$w'_{t+1} = (1+r)w'_t - P_{t-m} + R_{t-n}, \tag{31}$$

$$\mathbf{R}_{t+1} = (\mathbf{R}_t^{-1}, pD_t). \tag{32}$$

More specifically, a cost lower bound of the exact system with a longer collection period can be obtained by solving the following dynamic program: $V_{T+1}(x, \underline{w}) = 0$, and

$$V_t(x, \underline{w}) = \min_{y \geq x} \Big\{ G_t(y) + \nu_t(cy, \underline{w}) + \mathsf{E}V_{t+1}\Big[y - D_t, (1+r)\underline{w} + R_{t-n+m} - cD_t\Big]\Big\}. \tag{33}$$

Similarly, the cost lower bound of the exact system with a longer payment period can be obtained by solving the following dynamic program:

$$\begin{aligned} \bar{V}_t(x, \bar{w}) =& \min_{y \geq x} \Big\{ G_t(y) + M_t^-(cy - \bar{w} + p\mu^{m'}) \\ &+ \mathsf{E}\bar{V}_{t+1}\Big[y - D_t, (1+r)\bar{w} + (p-c)D_t + p\mu_{t+m'} - p\mu_t\Big]\Big\}. \end{aligned} \tag{34}$$

We can show that the (d, S) policy is optimal for the dynamic programs in (33) and (34) by a extending the proof of Proposition 4. The lower bound cost is then the optimal cost solved from either of the dynamic programs.

We conduct a simulation study to test the performance by comparing the (d, S) and $(\mathbf{d}, \mathbf{a}, S)$ heuristics with the lower bound solutions obtained from (33) and (34), respectively. We summarize the overall performance for $m, n = \{1, 2, 3\}$. In the combined test, let C be the cost of the heuristic policy and $\underline{C}$ be the cost of the lower bound system. The percentage error is defined as

$$\%\text{ error} = \frac{C - \underline{C}}{\underline{C}} \times 100\%.$$

The time horizon is $T = 8$ periods. Demand D_t is normally distributed with the first period mean $\mu_1 = 10$, and μ_t increasing at a rate of $\delta \in \{5\%, 10\%\}$ per period. We fix parameter $c = 1$ and set $p = 1 + \delta$, and vary the other parameters with each taking the values in the set: $h \in \{5\%, 10\%, 15\%\}$, $e \in \{1.5\%, 2\%, 2.5\%\}$, $r \in \{0.4\%, 0.8\%, 1.2\%\}$, $\sigma \in \{2, 3, 4\}$. The total number of instances generated for each heuristic is 1458. The average (maximum, minimum) performance error is 2.4% (6.7%, 0.2%). We observe that the performance deteriorates when the cash-rated cost parameters e and r are larger. Nevertheless, the heuristic performs well in general, as there are only 44 instances whose errors are greater than 5%. This is because the inventory and cash-related costs are minimized through dynamic program, and excluding them from the cash dynamics does not significantly affect the optimal cost. Note that this percentage error is calculated by comparing to the lower bound of the exact system. Clearly, the heuristic will perform better if comparing to the optimal cost.

General Demand Case

We further examine the effectiveness of the (d, S) and $(\mathbf{d}, \mathbf{a}, S)$ policies under general non-stationary demand. Unfortunately, we are not able to derive a cost lower bound to the exact system defined in (10) with general nonstationary demand. Therefore, we focus on the simplified models defined in (14) and (21), and develop a cost lower bound for both models. The lower bound, defined as C_L, is generated by assuming that the firm can return excess on-hand inventory at the purchasing cost in each period. Thus, the optimal solution can be obtained by solving T separable single-period problems. (Mathematically, this is equivalent to omitting the constraint $x_t \leq y_t$ in each period. The proof is available from the authors.)

We conduct another simulation study for different non-stationary demand forms and summarize the overall heuristic performance. In the test for each model, let C_U be the cost of the heuristic. We define the percentage error as

$$\% \text{ error} = \frac{C_U - C_L}{C_L} \times 100\%.$$

We consider two demand forms with negative shocks: seasonal demand and product life cycle demand. In each demand form, D_t is normally distributed with mean μ_t shown in Table 2. The values of other parameters remain the same as above. In total, we generate 128 instances.

Period (t)	1	2	3	4	5	6	7	8	9	10	11	12
μ_t - seasonal demand	10	12	20	60	20	12	10	12	20	60	20	12
μ_t - life cycle demand	10	12	14	18	22	38	50	56	60	52	36	8

Table 2: Demand mean under different non-stationary demand forms.

The average (maximum, minimum) performance error for the test bed instances is 0.76% (3.28%, 0.12%). When negative shocks exist in the demand sequence, the underlined heuristics perform well in general. Nevertheless, the heuristics perform less effectively when demand variability is large. To see this, recall from Proposition 4 that the myopic policy is optimal under the condition that the state variables (x, w) will stay within the band if they are in the band in each period. When the demand decreases, the optimal base-stock level will decrease accordingly. Therefore, it is probable that a small demand realization will cause the states to traverse outside the band, making the heuristic less effective. This is more likely to occur when demand is more variable.

6.2 Value of the Financial Information

Recall that the optimal (d, S) policy generalizes the classic base-stock policy when e is small (or ample cash supply) and the cash-constrained base-stock policy when e is large. In this subsection, we assess the value of the (d, S) policy over these two policies through a numerical study. Notice that the cost difference between the (d, S) policy and the classic base-stock policy can be viewed as the value of considering financial flows when making the inventory decision.

We focus the case of $m = n = 1$ and compute the percentage cost increase of each policy over the optimal (d, S) policy under different default penalty costs. We set $\sigma_t = 2$, $c = 1$, $p = 1.075$, $h = 4\%$, $r = 0.5\%$, and keep other parameters the same as in §6.1. We vary the default penalty cost rate from 1% to 3.5%.

As shown in Figure 4, the performance of the classic base-stock policy is fairly effective when penalty e is close to r, but becomes less effective when e increases. Recall from (17) that when e approaches r, the default threshold d and the base-stock level S are close to each other. Thus, the (d, S) policy behaves similarly as the base-stock policy. On the contrary, when e is large, d becomes smaller, meaning that the system should order up to either d or $\underline{w}$ instead of S under the base-stock policy. This result shows the importance of intra-departmental collaboration for the entrepreneurial firms as they are typically falling short of cash and subject to high default penalty costs.

Conversely, the performance of cash-constrained base-stock policy is quite effective under a large e, but becomes less effective when e decreases. This is because when e is large, d becomes small. The firm would order up to the system working capital $\underline{w}$ more frequently, making the (d, S) policy similar to the cash-constrained policy.

6.3 Impact of Payment Periods

Although the firm may benefit from a longer payment period, the supplier will usually quote a higher unit wholesale price to compensate for the postponed cash inflow. This price increase can be viewed as the implicit cost of trade credit for the firm. To fully understand this tradeoff, the firm needs to analyze the cost saving from requesting an extension of the payment period. In this subsection, we conduct a numerical study to illustrate the impact of the payment period on the firm's total cost. This result can be used for a decision support tool when bargaining a lower purchase price under a given payment period.

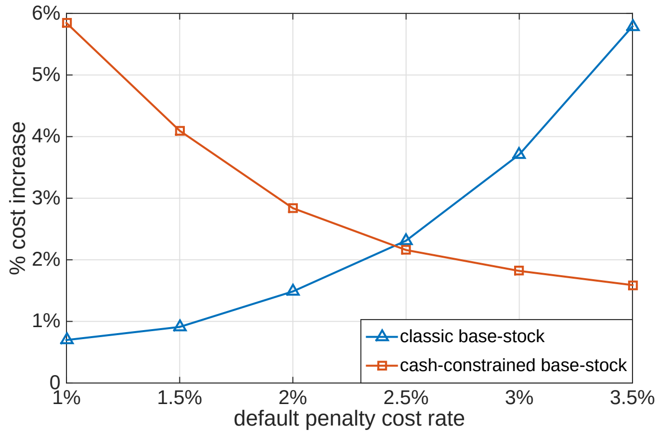


Figure 4: Impact of default penalty on the value of the optimal (d, S) policy

We compute the percentage cost reduction achieved by increasing the payment period m from 0 to 1 under different collection periods $n \in \{0, 1\}$.[8] (Intuitively, extending the payment period m will reduce the system cost; we shall provide a detailed discussion later.) To quantify the cost reduction, we keep the unit purchasing cost c unchanged. We use the (d, S) policy to compute the system cost for the model with a longer collection period and the $(\mathbf{d}, \mathbf{a}, S)$ policy for the model with a longer payment period.

We set $h = 5\%$, $e = 2.5\%$, $r = 1\%$, $p = 1.05$. Demand is normally distributed with mean in the first period $\mu_1 = 10$ and μ_t increasing at a constant rate of 10% per period. The rest of the parameters are the same as in §6.1. Figure 5 plots the percentage cost reduction curves with respect to demand volatility for $n \in \{0, 1\}$.

As shown in Figure 5, the cost reduction is positive when extending the payment period m (from 0 to 1) for any collection period n. This is because the firm can accumulate one more period of cash to pay for the inventory ordered, which decreases the chance of payment defaults. This observation is consistent with an empirical finding in Long et al. (1993), in which the authors find that suppliers with variable demand extend more trade credit than those with stable demand. (Extending payment period provides greater benefit to their buyers when the demand is more variable.) Notice that the cost reduction is not monotone when $n = 0$ when demand standard deviation σ_t increases from 2.5 to 4. This is because the benefit of cost reduction is undermined by the high demand volatility when the payment period is longer than the collection period.

Interestingly, the cost reduction of extending the payment period is higher when $n = 1$ than $n = 0$. This suggests that firms with a longer sales collection period have a stronger incentive to request their suppliers an extension of the payment period. Thus, we shall expect that the

[8] Here, one can view $n = 1$ as a net 30 contract and $n = 0$ as cash payment, i.e., the period length is one month.

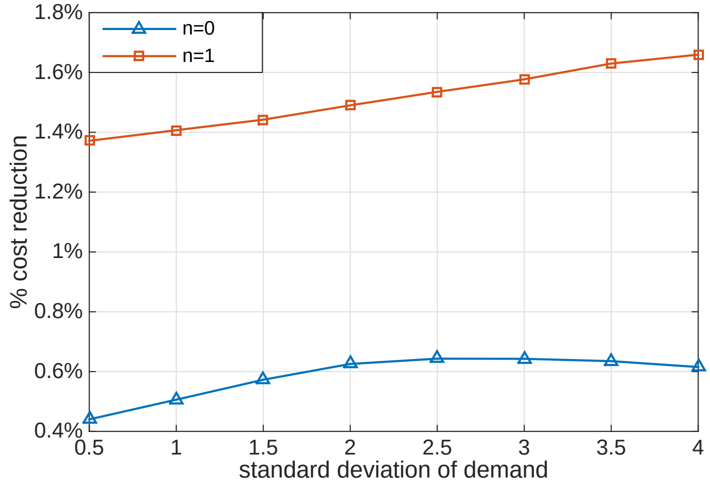


Figure 5: Impact of demand volatility on the cost reduction through payment period extension

upstream and downstream credit periods of a firm are positively correlated, which is consistent with the empirical findings in Guedes and Mateus (2009). To explain this, note that the firm is more likely to have a cash shortage when $n = 1$ than $n = 0$ as the customer payment is delayed for one period in the former case. This is more likely to happen especially when the demand is increasing. Thus, the benefit of extending the payment to the supplier is greater.

The above analysis of showing the benefit of cost reduction is based on the assumption that the purchase price c is constant. However, as stated, the supplier may increase the purchase price when the firm requests a payment extension. Thus, it is of interest to learn what the maximum acceptable purchase price is so that the firm is indifferent between different payment periods. Figure 6 shows these indifferent points (isocost) when extending the payment period m from 0 to 1 under different demand increasing rate for $n \in \{0, 1\}$. As shown in the figure, under a more blooming market, the benefit of payment extension is larger, which allows a higher increase in the purchase price. This provides the manager guidance to bargain a reasonable purchase price when negotiating a trade credit contract. For example, if a firm has the demand rate of 10% and the cash collection period is one month ($n = 1$), then the additional purchase price should not exceed 1.8% when it requests a one-month payment extension from the supplier.

7 Conclusion

This paper studies the impact of two-level trade credit on an entrepreneurial firm's inventory decision. We introduce a notion of working capital that simplifies the computation and characterizes the optimal and near-optimal policies. The resulting policies resemble the business practice of work-

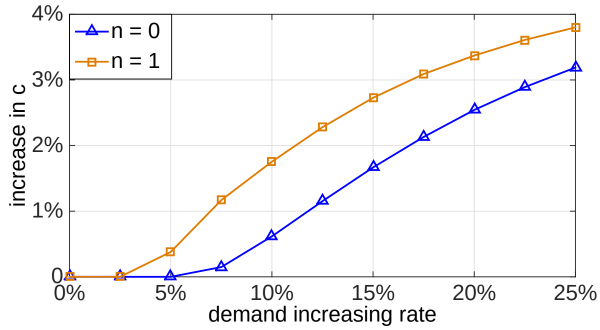


Figure 6: Isocost curves for purchase price premiums and demand rates under $n \in \{0, 1\}$

ing capital management in that firms review their working capital status before making inventory decisions. Our study reveals insights on the importance of collaboration between operations and accoutning/finance departments for entrepreneurial firms. In addition, our model generalizes the classic base-stock and the cash-constrained models. We believe that this generality reflects the real-world practice. The policy control parameters have a closed-form expression, which facilitates interdisciplinary teaching for students and practitioners. Finally, we demonstrate that the bullwhip and reverse bullwhip may be caused by customer payment defaults and how supplier's liquidity provisions can mitigate these effects.

A possible future work is to incorporate demand forecasting in the current model. Aviv (2007) demonstrates the benefit of sharing the forecast demand information with the upstream supplier. In this joint material and cash flow model, the demand forecast information will be translated into cash flow information. It will be of interest to investigate the value of demand and cash flow information for firms.

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Appendix A Optimality of the $(\mathbf{d}, \mathbf{a}, S)$ Policy

We show the optimality of the $(\mathbf{d}, \mathbf{a}, S)$ policy for the three-piece lower bound system. Let us first revisit the myopic solution. Define $a_t = a_t' + a_t''$. To solve the problem in (28), we consider the following five cases.

Case 1. When $\bar{w} \le cd_t - a_t''$, it is optimal to order up to threshold d_t, and $\bar{v}_t(\bar{w}) = \bar{L}_t(\bar{w} + a_t) + \bar{p}_t a_t$, where $\bar{L}_t(\bar{w}) = G_t(d_t + a_t'/c) - e(\bar{w} - cd_t - a_t')$.

Case 2. When $cd_t - a_t'' < \bar{w} \le c\bar{d}_t - a_t''$, it is optimal to default by a_t'' on expectation. In this case, $\bar{v}_t(\bar{w}) = G_t[(\bar{w} + a_t)/c] + \bar{p}_t a_t$.

Case 3. When $c\bar{d}_t - a_t'' < \bar{w} \le c\bar{d}_t + a_t'$, it is optimal to order up to threshold $\bar{d}_t$, in which case $\bar{v}_t(\bar{w}) = \bar{\bar{L}}_t(\bar{w}) = G_t(d_t + a_t'/c) - \bar{p}_t(\bar{w} - c\bar{d}_t - a_t')$.

Case 4. When $c\bar{d}_t + a_t' < \bar{w} \le cS_t + a_t'$, the system is working capital constrained. It is optimal to order up to $(\bar{w} - a_t')/c$ and leave no cash on hand on expectation. Thus, $\bar{v}_t(\bar{w}) = G_t[(\bar{w} - a_t')/c]$.

Case 5. When $cS_t + a_t' < \bar{w}$, the system has ample working capital and orders up to the target base stock S_t. In this case, the expected cash balance will be nonnegative, and $\bar{v}_t(\bar{w}) = G_t(S_t)$.

As a result, Equation (28) becomes

$$\bar{v}_t(\bar{w}) = \begin{Bmatrix} \bar{L}_t(\bar{w}+a_t)+\bar{p}_t a_t, & \text{if } \bar{w} \le cd_t - a''_t \\ G_t[(\bar{w}+a_t)/c]+\bar{p}_t a_t, & \text{if } cd_t - a''_t < \bar{w} \le c\bar{d}_t - a''_t \\ \bar{\bar{L}}_t(\bar{w}), & \text{if } c\bar{d}_t - a''_t < \bar{w} \le c\bar{d}_t + a'_t \\ G_t[(\bar{w}-a'_t)/c], & \text{if } c\bar{d}_t + a'_t < \bar{w} \le cS_t + a'_t \\ G_t(S_t), & \text{if } cS_t + a'_t < \bar{w} \end{Bmatrix}. \tag{35}$$

Similar to (20) we define the "band" as

$$\bar{\mathcal{B}}_t = \left\{(x_t, \bar{w}_t) \in \Re^2 \mid x_t \le \bar{y}^*_t(\bar{w}_t)\right\}.$$

Let us define $A_t = F^{m'}_t(\mu^{m'}_t)$ as a measure of asymmetry of demand D^m_t. In analogy to Proposition 4, the following proposition shows the optimality through decoupling.

Proposition 9. *Assume that (1) A_t is non-increasing in t; (2) both A'_t and A''_t are non-decreasing in t. Then we have:*

(a) The control parameters d_t, $\bar{d}_t$ and S_t are non-decreasing in t, and $d_t \le \bar{d}_t \le S_t$ for all t;

(b) $\bar{V}_t(x,\bar{w}) = \bar{W}_t(\bar{w})$ for all t and $(x,\bar{w}) \in \bar{\mathcal{B}}_t$, where

$$\bar{W}_t(\bar{w}) = \bar{v}_t(\bar{w}) + \mathsf{E}\bar{W}_{t+1}(\bar{w} + (p-c)D_t + p\mu_{t+m'} - p\mu_t),$$

and $\bar{W}_{T+1}(\bar{w}) = 0$; $\bar{W}_t(\bar{w})$ is convex in $\bar{w}$;

(c) The $(\mathbf{d}, \mathbf{a}, S)$ policy is optimal for three-piece lower bound system in (26).

Assumption (1) requires, typically but not necessarily, that the aggregated demand $D^{m'}_t$ is less right-skewed when t gets larger. Note that most of the real life demand functions, such as Poisson(λ) and Gamma(k, 1), are right-skewed and become more symmetric under larger mean values (λ and k), hence satisfying Assumption (1). For zero-skewed (or symmetric) distributions, such as Normal, the following lemma guarantees Assumption (1) and (2). Moreover, most asymmetric demand distributions (Poisson, Gamma, etc.) can be shown or tested to satisfy Assumption (2).

The following Lemma implies when A_t constant over t, Assumptions (2) will always be satisfied.

Proposition 10. *If A_t is constant over t, then both A'_t and A''_t are non-decreasing in t.*

Appendix B Selected Proofs

Proposition 1.

Proof. We prove by induction. Clearly $\hat{V}_{T+1}$ is jointly convex in $(x_{T+1}, w'_{T+1}, \mathbf{P}_{T+1}, \mathbf{R}_{T+1})$. Assume that $\hat{V}_{t+1}$ is jointly convex in $(x_{t+1}, w'_{t+1}, \mathbf{P}_{t+1}, \mathbf{R}_{t+1})$. We show the property for t. Let us define

$$\hat{y}_t = \lambda y_t + (1-\lambda)\bar{y}_t,$$

and in the same way for $\hat{x}_t$, $\hat{x}_t$, $\hat{w}'_t$, $\hat{\mathbf{P}}_t$, and $\hat{\mathbf{R}}_t$. We first prove $\hat{V}_{t+1}(x_{t+1}, w'_{t+1}, \mathbf{P}_{t+1}, \mathbf{R}_{t+1})$ is jointly convex in $(y_t, x_t, w'_t, \mathbf{P}_t, \mathbf{R}_t)$, with the state dynamics specified in (2)-(5). Note that for any period t and fixed $(x_t, \mathbf{P}_t, \mathbf{R}_t)$, $\hat{V}_t(x_t, w'_t, \mathbf{P}_t, \mathbf{R}_t)$ is decreasing in w'_t. This is intuitive as the more starting cash the firm has, the lower total expected costs it incurs. Similarly, it can be easily seen that for any period t and fixed $(x_t, w'_t, \mathbf{P}_t, \mathbf{R}_t^{-1})$, $\hat{V}_t(x_t, w'_t, \mathbf{P}_t, \mathbf{R}_t)$ is decreasing in R_{t-1}. Also note that $r(P_{t-m} - w'_t)^- = r(P_{t-m} - w'_t)^+ - r(P_{t-m} - w'_t)$, and $p\min\{y_t, D_t\} = py_t - p(y_t - D_t)^+$. Substituting these into the state dynamics, we have for any $(y_t, x_t, w'_t, \mathbf{P}_t, \mathbf{R}_t)$, $(\bar{y}_t, \bar{x}_t, \bar{w}'_t, \bar{\mathbf{P}}_t, \bar{\mathbf{R}}_t)$, and $0 \le \lambda \le 1$,

$$
\begin{aligned}
&\hat{V}_{t+1}(\hat{y}_t - D_t, (1+r)\hat{w}'_t + \hat{R}_{t-n} - (1+r)\hat{P}_{t-m} - h(\hat{y}_t - D_t)^+ - (e-r)(\hat{P}_{t-m} - \hat{w}'_t)^+ \\
&\quad - r(\hat{P}_{t-m} - \hat{w}'_t), \hat{\mathbf{P}}_t^{-1}, c(\hat{y}_t - \hat{x}_t), \hat{\mathbf{R}}_t^{-1}, p\hat{y}_t - p(\hat{y}_t - D_t)^+) \\
&\le \hat{V}_{t+1}(\hat{y}_t - D_t, (1+r)\hat{w}'_t + \hat{R}_{t-n} - (1+r)\hat{P}_{t-m} - h\lambda(y_t - D_t)^+ - h(1-\lambda)(\bar{y}_t - D_t)^+ \\
&\quad - (e-r)\lambda(P_{t-m} - w'_t)^+ - (e-r)(1-\lambda)(\bar{P}_{t-m} - \bar{w}'_t)^+ - r(\hat{P}_{t-m} - \hat{w}'_t), \\
&\quad \hat{\mathbf{P}}_t^{-1}, c(\hat{y}_t - \hat{x}_t), \hat{\mathbf{R}}_t^{-1}, p\hat{y}_t - p\lambda(y_t - D_t)^+ - p(1-\lambda)(\bar{y}_t - D_t)^+) \qquad (36) \\
&\le \lambda\hat{V}_{t+1}(y_t - D_t, (1+r)w'_t + R_{t-n} - (1+r)P_{t-m} - h(y_t - D_t)^+ - (e-r)(P_{t-m} - w'_t)^+ \\
&\quad - r(P_{t-m} - w'_t), \mathbf{P}_t^{-1}, c(y_t - x_t), \mathbf{R}_t^{-1}, py_t - p(y_t - D_t)^+) \\
&+ (1-\lambda)\hat{V}_{t+1}(\bar{y}_t - D_t, (1+r)\bar{w}'_t + \bar{R}_{t-n} - (1+r)\bar{P}_{t-m} - h(\bar{y}_t - D_t)^+ - (e-r)(\bar{P}_{t-m} - \bar{w}'_t)^+ \\
&\quad - r(\bar{P}_{t-m} - \bar{w}'_t), \bar{\mathbf{P}}_t^{-1}, c(\bar{y}_t - \bar{x}_t), \bar{\mathbf{R}}_t^{-1}, p\bar{y}_t - p(\bar{y}_t - D_t)^+), \qquad (37)
\end{aligned}
$$

where the inequality in (36) is due to the above mentioned monotonicity result (i.e., $\hat{V}_t(x_t, w'_t, \mathbf{P}_t, \mathbf{R}_t)$ decreases with w'_t and R_{t-1}) and the convexity of functions $(y_t - D_t)^+$ and $(P_{t-m} - w'_t)^+$, i.e.,

$$
\begin{aligned}
(\hat{y}_t - D_t)^+ &\le \lambda(y_t - D_t)^+ + (1-\lambda)(\bar{y}_t - D_t)^+, \\
(\hat{P}_{t-m} - \hat{w}'_t)^+ &\le \lambda(P_{t-m} - w'_t)^+ + (1-\lambda)(\bar{P}_{t-m} - \bar{w}'_t)^+.
\end{aligned}
$$

And the inequality in (37) follows from the joint convexity of $\hat{V}_{t+1}$, due to induction.

Therefore, $\hat{V}_{t+1}(x_{t+1}, w'_{t+1}, \mathbf{P}_{t+1}, \mathbf{R}_{t+1})$ is jointly convex in $(y_t, x_t, w'_t, \mathbf{P}_t, \mathbf{R}_t)$, and so is its expected value. Furthermore, it can be easily shown that $\Big(g_t(y_t) + \nu_t(P_{t-m}, w'_t)\Big)$ is also jointly convex, and the constraint $y_t \ge x_t$ forms a convex set. Applying Proposition B-4 of Heyman and Sobel (1984) we conclude that $\hat{V}_t$ is jointly convex in $(x_t, w'_t, \mathbf{P}_t, \mathbf{R}_t)$, completing the induction. □

Proposition 3.

Proof. We define $\pi_t(y, \underline{w}) = G_t(y) + e(cy - \underline{w})^+ - r(cy - \underline{w})^-$ and take the partial derivative with respect to y:

$$
\frac{\partial}{\partial y}\pi_t(y, \underline{w}) = \frac{\partial}{\partial y}G_t(y) + \left\{ \begin{array}{ll} ec, & \text{if } cy > \underline{w} \\ 0, & \text{if } cy = \underline{w} \\ rc, & \text{if } cy < \underline{w} \end{array} \right\}. \qquad (38)
$$

Now, let us consider the three cases in the (d, S) policy. For Case 1, i.e., $\underline{w} \le cd_t$, it can be shown from (16) that for small positive ϵ, $\frac{\partial}{\partial y}\pi_t(d_t - \epsilon, \underline{w}) < 0$ and $\frac{\partial}{\partial y}\pi_t(d_t + \epsilon, \underline{w}) > 0$. Since $\pi_t(y, \underline{w})$ is convex in y, we have $y_t^*(\underline{w}) = d_t$ when $\underline{w} \leq d_t$. The other two cases can be similarly proved. $\square$

Proposition 4.

Proof. (a) can be directly obtained from (17) and the definition of the usual stochastic order. We prove (b) and (c) by induction. The claim trivially holds for $t = T + 1$. Assume $V_{t+1}(x_t, \underline{w}_t) = W_{t+1}(\underline{w}_t)$ for all $(x_t, \underline{w}_t) \in \mathcal{B}_{t+1}$, then

$$V_t(x_t, \underline{w}_t) = \min_{y_t \geq x_t} \left\{ J_t(y_t, \underline{w}_t) \right\}, \tag{39}$$

where $J_t(y_t, \underline{w}_t) = G_t(y_t) + e(cy_t - \underline{w}_t)^+ - r(cy_t - \underline{w}_t)^- + \mathsf{E}V_{t+1}(y_t - D_t, \underline{w}_t + R_{t-n+m} - cD_t)$. To solve the problem in (39), we consider the following three cases.

Case 1: $\underline{w}_t \leq cd_t$. To see that d_t is a minimizer of J_t, note from (a) and demand non-negativity that $x_{t+1} = d_t - D_t \leq d_{t+1}$, i.e., $(x_{t+1}, \underline{w}_{t+1}) \in \mathcal{B}_{t+1}$. From induction and Proposition 3, it can be shown that $y_t^*(\underline{w}_t) = d_t$. If $x_t \leq d_t$, i.e., $(x_t, \underline{w}_t) \in \mathcal{B}_t$, the base-stock is achievable, then

$$W_t(\underline{w}_t) = \min_{x_t \leq y_t} \{J_t(y_t, \underline{w}_t)\} = \mathsf{E}W_{t+1}(\underline{w}_t + R_{t-n+m} - cD_t) + L_t(\underline{w}_t).$$

Case 2: $cd_t < \underline{w}_t \leq cS_t$. To see that $\underline{w}_t/c$ is a minimizer of J_t, note from (a) and demand non-negativity that $x_{t+1} = \underline{w}_t/c - D_t \leq S_t \leq S_{t+1}$ and $x_{t+1} = \underline{w}_t/c - D_t \leq \underline{w}_t/c + R_{t-n+m}/c - D_t = \underline{w}_{t+1}/c$. Therefore, $x_{t+1} \leq S_{t+1} \wedge \underline{w}_{t+1}/c$, i.e., $(x_{t+1}, \underline{w}_{t+1}) \in \mathcal{B}_{t+1}$. From induction and Proposition 3 we have $y_t^*(\underline{w}_t) = \underline{w}_t/c$. If $x_t \leq \underline{w}_t/c$, i.e., $(x_t, \underline{w}_t) \in \mathcal{B}_t$, the base-stock is achievable, then

$$W_t(\underline{w}_t) = \min_{x_t \leq y_t} \{J_t(y_t, \underline{w}_t)\} = \mathsf{E}W_{t+1}(\underline{w}_t + R_{t-n+m} - cD_t) + G_t(\underline{w}_t/c).$$

Case 3: $cS_t < \underline{w}_t$. To see that S_t is a minimizer of J_t, note from (a) and demand non-negativity that $x_{t+1} = S_t - D_t \leq S_{t+1}$ and $x_{t+1} = S_t - D_t < \underline{w}_t/c + R_{t-n+m}/c - D_t = \underline{w}_{t+1}$. Therefore, $x_{t+1} \leq S_{t+1} \wedge \underline{w}_{t+1}/c$, i.e., $(x_{t+1}, \underline{w}_{t+1}) \in \mathcal{B}_{t+1}$. From induction and Proposition 3, it can be shown that $y_t^*(\underline{w}_t) = S_t$. If $x_t \leq S_t$, i.e., $(x_t, \underline{w}_t) \in \mathcal{B}_t$, the base-stock is achievable, then

$$W_t(\underline{w}_t) = \min_{x_t \leq y_t} \{J_t(y_t, \underline{w}_t)\} = \mathsf{E}W_{t+1}(\underline{w}_t + R_{t-n+m} - cD_t) + R_t(S_t).$$

Summarizing the above three cases, we prove the optimality of the (d, S) policy and the decomposition of $V_t(x, w)$. Moreover, since $W_{t+1}(\cdot)$ is convex from induction, $W_t(\cdot)$ is also convex. $\square$

Proposition 6.

Proof. Given the expression of $M(u)$, we take derivative with respect to u as below.

$$\frac{\partial M(u)}{\partial u} = eF^{m'}(u/p) + r\bar{F}^{m'}(u/p) = (e - r)F^{m'}(u/p) + r.$$

Set $u = p\mu^{m'}$, we have $\bar{p} = (e-r)F^{m'}(\mu^{m'}) + r$, and $M(p\mu^{m'}) = (e-r)p\hat{F}^{m'}(\mu^{m'})$. Hence,

$$\Gamma(u) = \Big[(e-r)F^{m'}(\mu^{m'}) + r\Big](u - p\mu^{m'}) + (e-r)p\hat{F}^{m'}(\mu^{m'}),$$

from which the expressions of a' and a'' immediately follow. □

Proposition 8.

Proof. We define $\bar{\pi}_t(y,\bar{w}) = G_t(y) + \bar{M}_t(cy - \bar{w} + p\mu_t^{m'})$ and take derivative with respect to y:

$$\frac{\partial}{\partial y}\bar{\pi}_t(y,\bar{w}) = \frac{\partial}{\partial y}G_t(y) + \left\{\begin{array}{ll} r, & \text{if } cy < \bar{w} - a'_t \\ \bar{p}_t, & \text{if } \bar{w} - a'_t < cy < \bar{w} + a''_t \\ e, & \text{if } \bar{w} + a''_t < cy \end{array}\right\}. \tag{40}$$

Now, let us consider the five cases in the $(\mathbf{d}, \mathbf{a}, S)$ policy. For Case 1, i.e., $\bar{w} \le cd_t - a''_t$, it can be shown from (29) that for small positive ϵ, $\frac{\partial}{\partial y}\bar{\pi}_t(d_t - \epsilon, \bar{w}) < 0$ and $\frac{\partial}{\partial y}\bar{\pi}_t(d_t + \epsilon, \bar{w}) > 0$. Since $\bar{\pi}_t(y,\bar{w})$ is convex in y, we have $\bar{y}_t^*(\bar{w}) = d_t$ when $\bar{w} \le cd_t - a''_t$. The proofs of other cases are similar. □

Proposition 9.

Proof. (a) can be directly obtained from (16), (29), and the definition of the usual stochastic order. We prove (b) and (c) by induction. Let us derive $y^*(w)$ from (27) and the definition of $\bar{w}$ as follows:

$$y^*(w) = \left\{\begin{array}{ll} d, & \text{if } w + pA'' \le cd \\ (w + pA'')/c, & \text{if } cd < w + pA'' \le c\bar{d} \\ \bar{d}, & \text{if } w + pA' \le c\bar{d} < w + pA'' \\ (w + pA')/c, & \text{if } c\bar{d} < w + pA' \le cS \\ S, & \text{if } cS < w + pA' \end{array}\right\}. \tag{41}$$

Thus, $x_t \le y_t^*(w_t)$ is equivalent to $x_t \le \bar{y}_t^*(\bar{w}_t)$, i.e., $(x_t, \bar{w}_t) \in \bar{\mathcal{B}}_t$. The claim trivially holds for $t = T+1$. Assume $\bar{V}_{t+1}(x,\bar{w}) = \bar{W}_{t+1}(\bar{w})$ for all $(x,\bar{w}) \in \bar{\mathcal{B}}_{t+1}$, then

$$\bar{V}_t(x,\bar{w}) = \min_{y \ge x}\left\{\bar{J}_t(y,\bar{w})\right\}, \tag{42}$$

where $\bar{J}_t(y,\bar{w}) = G_t(y) + \bar{M}_t(cy - \bar{w} + p\mu_t^{m'}) + \mathsf{E}\bar{V}_{t+1}(y - D_t, \bar{w} + (p-c)D_t + p\mu_{t+m'} - p\mu_t)$. To solve the problem in (42), we consider the following five cases.

Case 1: $\bar{w}_t \le cd_t - a''_t$, i.e., $w_t + pA''_t \le cd_t$. To see that d_t is a minimizer of $\bar{J}_t$, note from (a) and demand non-negativity that $x_{t+1} = d_t - D_t \le d_{t+1}$. Therefore, we have $(x_{t+1}, \bar{w}_{t+1}) \in \bar{\mathcal{B}}_{t+1}$. From induction and Proposition 8, it can be shown that $\bar{y}_t^*(\bar{w}_t) = d_t$. If $x_t \le d_t$, i.e., $(x_t, \bar{w}_t) \in \bar{\mathcal{B}}_t$, the base-stock is achievable, the rest of the proof is similar to Proposition 4.

Case 2: $cd_t - a''_t < \bar{w}_t \le c\bar{d}_t - a''_t$, i.e., $cd_t < w_t + pA''_t \le c\bar{d}_t$. To see that $(\bar{w}_t + a''_t)/c$ is a minimizer of $\bar{J}_t$, note from (a) and demand non-negativity that $x_{t+1} = (w_t + pA''_t)/c - D_t \le \bar{d}_{t+1}$ and $x_{t+1} = (w_t + pA''_t)/c - D_t \le (w_t + (p-c)D_t + pA''_t)/c \le (w_{t+1} + pA''_{t+1})/c$. Therefore, we have $(x_{t+1}, \bar{w}_{t+1}) \in \bar{\mathcal{B}}_{t+1}$. The rest of the proof is similar to Case 1.

Case 3: $c\bar{d}_t - a''_t < \bar{w}_t \le c\bar{d}_t + a'_t$, i.e., $w_t + pA'_t \le c\bar{d}_t < w_t + pA''_t$. To see that $\bar{d}_t$ is a minimizer of $\bar{J}_t$, note from (a) and demand non-negativity that $x_{t+1} = \bar{d}_t - D_t \le \bar{d}_{t+1}$ and $x_{t+1} = \bar{d}_t - D_t < (w_t + (p-c)D_t + pA''_t)/c \le (w_{t+1} + pA''_{t+1})/c$. Therefore, we have $(x_{t+1}, \bar{w}_{t+1}) \in \bar{\mathcal{B}}_{t+1}$. The rest of the proof is similar to Case 1.

Case 4: $c\bar{d}_t + a'_t < \bar{w}_t \le cS_t + a'_t$, i.e., $c\bar{d}_t < w_t + pA'_t \le cS_t$. To see that $(\bar{w}_t - a'_t)/c$ is a minimizer of $\bar{J}_t$, note from (a) and demand non-negativity that $x_{t+1} = (w_t + pA'_t)/c - D_t \le S_{t+1}$ and $x_{t+1} = (w_t + pA'_t)/c - D_t \le (w_t + (p-c)D_t + pA'_t)/c \le (w_{t+1} + pA'_{t+1})/c$. Therefore, we have $(x_{t+1}, \bar{w}_{t+1}) \in \bar{\mathcal{B}}_{t+1}$. The rest of the proof is similar to Case 1.

Case 5: $cS_t + a'_t < \bar{w}_t$, i.e., $cS_t < w_t + pA'_t$. To see that S_t is a minimizer of $\bar{J}_t$, note from (a) and demand non-negativity that $x_{t+1} = S_t - D_t \le S_{t+1}$ and $x_{t+1} = S_t - D_t \le (w_t + (p-c)D_t + pA'_t)/c \le (w_{t+1} + pA'_{t+1})/c$. Therefore, from (41) we have $x_{t+1} \le y^*_t(w_{t+1})$, thus, $(x_{t+1}, \bar{w}_{t+1}) \in \bar{\mathcal{B}}_{t+1}$. The rest of the proof is similar to Case 1.

Summarizing the above three cases, we prove the optimality of the $(\mathbf{d}, \mathbf{a}, S)$ policy and the decomposition of $\bar{V}_t(x, \bar{w})$. Since $\bar{W}_{t+1}(\cdot)$ is convex from induction, $\bar{W}_t(\cdot)$ is also convex. □

Proposition 10.

Proof. To simplify the notation, we assume $m = 1$ and drop the superscript without loss of generalization. By definition of loss function, we have

$$\hat{F}_t(\mu_t) = \int_{\mu_t}^{\infty} \bar{F}_t(y)dy = \int_0^{\mu_t} F_t(y)dy = \int_0^{F_t(\mu_t)} (\mu_t - F_t^{-1}(y))dy,$$

where $F^{-1}(\cdot)$ is the inverse function of $F(\cdot)$. Hence we have

$$A'_t = (\mu_t F_t(\mu_t) - \hat{F}_t(\mu_t))/F_t(\mu_t) = \int_0^{F_t(\mu_t)} F_t^{-1}(y)dy/F_t(\mu_t).$$

Similarly, it can be shown that $A''_t = \int_{F_t(\mu_t)}^1 F_t^{-1}(y)dy/\bar{F}_t(\mu_t)$. Due to usual stochastic order, $F_t^{-1}(y)$ is non-decreasing in t for any $y \in [0,1]$. Given that $A_t = F_t(\mu_t) = 1 - \bar{F}_t(\mu_t)$ is constant over t, we have A'_t and A''_t are non-decreasing in t. □