Operational Investment and Capital Structure Under Asset Based Lending

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Abstract

We study the implications of asset based lending for operational investment, probability of bankruptcy, and capital structure for a borrower firm. We set up a single-period game with two players, a business owner and a bank. The business owner decides how to allocate her capital between the equity of a new business and the external capital market in order to maximize her expected profit. We model the new business as a single-period inventory (newsvendor) model. The bank does not know the newsvendor's demand distribution, and sets an asset based credit limit to maximize its expected profit. We show that the equilibrium order quantity is a function of market parameters, and deviates from the classical newsvendor solution. In this solution, asset based lending leads to an upper limit on the potential loss faced by the bank, and thus, helps manage bankruptcy risk. In particular, the collateral value of inventory is a function of the bank's belief regarding the firm's demand distribution because the amount of inventory that will have to be liquidated in case of a default is random and depends on the realized demand. This result contrasts with the common practice of banks to use simple rules of thumb to value inventory and set a credit limit. We also show that the probability of bankruptcy and the capital structure at equilibrium are functions of information asymmetry, bankruptcy costs, and the newsvendor model parameters.

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1 Introduction

Asset Based Lending (ABL) is a method commonly used by banks to lend money to small and risky businesses. In this method, a borrower firm offers its current assets, which include its inventory, cash, and account receivables, as collateral for a secured loan. The bank values the current assets and thereby sets a credit limit for the firm. Asset based loans secured solely by inventory are common in practice, especially in the retail industry (Foley et al. 2012; GE Capital 1999: page 14), which is one of the top three asset based borrowers (Commercial Finance Association 2009). One of the key decisions in ABL is the determination of the collateral value of inventory. Typically, banks use simple rules of thumb to set a collateral value, such as the type of the inventory and its age (GE Capital 1999). For example, an inventory of books will be valued differently from that of fashion clothing. However, a bank needs a more sophisticated model of the firm's operations when the demand for firm's products is uncertain. Whereas the collateral is based on the starting inventory, the bank's loan recovery in case of bankruptcy is random and depends on the realized demand, the amount of leftover inventory, and the inventory liquidation cost. Moreover, the computation of the credit limit affects the actions of the borrower firm. Thus, it determines the bank's profit and loan recovery, the probability of bankruptcy of the firm, the debt-equity mix in the capital structure of the firm, as well as the quantity of inventory procured. Examining these implications is valuable for banks, borrower firms, investors, as well as policy makers.

ABL is a large industry. In 2009, the total amount of outstanding asset based loans in the USA was $480 billion (Commercial Finance Association 2009), which constituted 25% of the total amount of loans and short term papers issued to nonfinancial corporations (Board of Governors of the Federal Reserve System 2010). ABL is useful to banks and borrowers alike because of its benefits with respect to information asymmetry and cost of lending. Banks are usually not well-informed regarding the future demand prospects of a borrower. ABL mitigates the cost of this information asymmetry by preventing over-borrowing. Furthermore, since an asset based loan is secured by the borrower's current assets, the bank can recover some of its losses by liquidating these assets in case of a default. Thus, ABL requires less monitoring and simpler financial covenants, and carries a lower interest rate than unsecured loans. For example, Foley et al. (2012) report that, in July

2011, Dick's Sporting Goods obtained a $440M credit line at the prime interest rate secured solely by inventory. They present many other examples illustrating the size of these loans in the retailing industry. The Small Business Association (SBA) also offers ABL programs, such as the Standard Asset Based CAPLines program (Godfrey 2011) and the Small Business Lending Fund established for community banks as a part of the Small Business Jobs Act (The Secured Lender Industry News 2011). These programs allow participating banks to provide asset based loans to small businesses, wherein the SBA sets a maximum allowable interest rate. For example, this interest rate was equal to the prime rate plus 2.25% as of November 2011 (SBA 2011). These features make ABL accessible to small businesses, which typically do not have access to cash flow financing availed by large companies with revenues in excess of $25 million and stable profits (Burroughs 2008).

This paper presents a model at the interface of operations and corporate finance to study the bank's optimal lending decision under ABL, and examine its implications for the operational investment, capital structure, and probability of bankruptcy of a borrower firm. We focus on loans given to small businesses secured solely by inventory. We set up a single-period game theoretic model with two players, a business owner and a bank. The business owner decides how to allocate her capital between the equity of a new business and the external capital market in order to maximize her expected profit. The business is represented using the newsvendor model. The owner sets up the business as a limited liability firm, interacts with the bank on its behalf, and manages its operations. The bank provides a loan to the firm at a fixed interest rate. It sets a credit limit, secured by inventory, to maximize its expected profit when it is partially informed about the demand distribution of the newsvendor firm.

We answer three research questions. First, how should the bank determine the collateral value of inventory to maximize its expected profit under information asymmetry and demand uncertainty? We show that the collateral value is higher than the salvage value of leftover inventory, and is not constant. It depends on the amount of inventory procured by the firm, various parameters of the inventory model, and the bank's belief regarding the firm's demand distribution and other parameters. This result contrasts with the common practice of banks to use simple rules of thumb to value inventory. Moreover, the credit limit based on this collateral value decomposes into two

components: a riskless component which can be recovered even if the realized demand is zero, and a risky component which is tied to the firm's demand prospects and information asymmetry. Thus, we identify the mechanism by which ABL mitigates the cost of adverse selection for the bank.

Second, what should be the equity investment and order quantity decisions of the owner at equilibrium? On the one hand, the owner's actions are guided by the opportunity cost of capital. The higher the rate of return in the external capital market, the lesser should be the equity investment of the owner in the newsvendor business, and the higher should be the reliance on debt to finance the inventory. On the other hand, the bank anticipates the owner's action and prevents the firm from over-borrowing by imposing an asset based credit limit. The resulting equilibrium outcome can be of three types: when the firm borrows and has non-zero probability of bankruptcy, when the firm borrows with zero probability of bankruptcy, and when the firm does not borrow. We present the equilibrium solution in all these cases and show when they occur.

Third, what is the probability of bankruptcy of the firm at equilibrium? Interestingly, we find that when the firm borrows with bankruptcy risk, the credit limit is *always* binding. In other words, a solution in which the firm may borrow with bankruptcy risk and not use up the entire credit limit never arises at equilibrium. Furthermore, the bank sets the optimal credit limit in such a way that the probability of bankruptcy becomes independent of the owner's choice of debt and equity as long as the firm borrows with risk. We derive the resulting formula for the probability of bankruptcy and show its dependence on the newsvendor model parameters.

Despite its practical usage, ABL has not been well studied in the academic literature. The traditional analytical models of capital structure in corporate finance focus on bondholders, and do not include either the setting of a credit limit or the details of operational decisions. In the recent years, there has been growing interest in joint operations-finance decision models. In particular, Buzacott and Zhang (2004) propose a model for ABL and provide many important insights. Our paper generalizes their model by incorporating the owner's equity decision, information asymmetry, taxation and cost of bankruptcy. It contributes to the operations-finance interface literature by being the first paper to model operational and capital structure decisions under information asymmetry. We show how to value inventory under ABL, and explain why ABL is an effective lending

mechanism under information asymmetry. We also show that firm characteristics, captured by the newsvendor model parameters, and the economic environment, captured by the return on the external investment alternative, the lending interest rate, and the lender's sentiment (i.e., the bank's belief regarding the newsvendor's demand distribution), affect the owner's operational and financial decisions and the probability of bankruptcy.

2 Literature Review

Our paper builds on the literature in corporate finance and the operations-finance interface. We describe relevant papers in brief.

Modigliani and Miller (1958) show that, in a perfect market, the capital structure of a firm is irrelevant to its optimal operational decisions. That is, the decision that maximizes the value to shareholders is equal to the decision that maximizes the total value of the firm. Subsequent research has led to two competing theories of capital structure, the tradeoff theory and the pecking order theory. Research on the tradeoff theory shows the existence of an optimal capital structure due to market frictions such as interest rate spread, taxation, costly bankruptcy, and liquidity constraints (e.g., Modigliani and Miller 1963, Kraus and Litzenberger 1973, Gordon 1989), but under *complete information*. A few papers in this stream consider firms' operational decision models in detail. In particular, Stiglitz (1972) shows a connection between operational and financial decisions under bankruptcy risk, and Dotan and Ravid (1985) model the optimal capacity, financing, and production decisions with uncertain sales price. In contrast to the tradeoff theory, the pecking order theory shows the existence of a financing hierarchy that minimizes the costs related to *incomplete information* (Jensen and Meckling 1976, Myers 1984, Myers and Majluf 1984, Childs et al. 2005). According to Frank and Goyal (2008), the pecking order theory models are relatively simple with linear objective functions, and thus, illustrate financing hierarchy under strong modeling assumptions, rather than giving a unifying framework.

The recent literature in capital structure is largely empirical. Interestingly, it obtains some findings that relate to operational characteristics of firms but cannot be explained by the theoretical models. For example, Lemmon et al. (2008) show that there is substantial unexplained variation

in capital structure, which is firm-specific and time-invariant; Rauh and Sufi (2010) illustrate that what a firm produces and the production assets it uses are the most important determinants of capital structure in the cross section; and Campello and Giambona (2010) show that firms with redeployable assets have more debt capacity. Other empirical papers have investigated the relationship of capital structure with various firm characteristics, including profitability, growth, liquidation value, return volatility, and operational risks. See Harris and Raviv (1991) and Leary and Graham (2010) for extensive reviews of the empirical capital structure literature.

Our paper builds on the capital structure literature by studying ABL under information asymmetry, incorporating the newsvendor model framework, and allowing an external investment option to the owner. This combination of a practical borrowing model and a realistic investment scenario reveals new insights regarding the interaction of the owner's operational and financial decisions. For example, our model yields predictions that are consistent with empirical observations in corporate finance with respect to the impact of operational characteristics, such as profitability, demand volatility, and asset recoverability (salvage value in our model), on a firm's capital structure. Additionally, contrary to the majority of the asymmetric information models that lead to underinvestment (Hubbard 1998), our paper shows a non-monotone relationship between information asymmetry and operational investment.

The operations management literature on joint operational-financial decisions addresses market imperfections by including taxes, liquidity constraints, bankruptcy risk, costly issuance of debt and equity, and credit limits into single- and multi-period inventory models. Among single-period models, Xu and Birge (2004) investigate the tradeoff between bankruptcy costs and the tax benefits of debt in a cash-constrained newsvendor model. Their analysis shows that integrating operational and financial decisions can improve firm value. Buzacott and Zhang (2004) study single- and multi-period models with asset based credit limit. The second half of their paper is relevant to our work. It analyzes a single-period model in which a newsvendor and a bank seek to maximize own profits in the presence of bankruptcy risk. Dada and Hu (2008) use a similar framework, with the difference that the bank chooses an optimal interest rate to charge to the newsvendor, instead of imposing a borrowing limit. They show the existence and uniqueness of an equilibrium order quantity-interest

rate pair. Multi-period inventory models analyze similar operational issues with additional financial dynamics, such as cash flows, dividend payments and capital subscriptions, e.g., Li et al. (2005), Hu and Sobel (2005), Chao et al. (2008), Hu et al. (2010).

Research on the impact of financial considerations on operational decisions is not limited to inventory models. Financial constraints and the risk of bankruptcy also affect the firm's survival strategy (Archibald et al. 2002), relations with its supply chain partners (Lai et al. 2009, Babich 2010, Kouvelis and Zhao 2010, Yang and Birge 2011), the choice of production technologies (Lederer and Singhal 1994, Boyabatli and Toktay 2010), the optimal time to shut down a firm (Xu and Birge 2006), and the optimal time to offer an IPO (Babich and Sobel 2004).

We contribute to the single-period operations management models by endogenizing the owner's equity decision. In our model, the owner faces a tradeoff between investing in the newsvendor and investing in the external capital market, which shapes the equity investment decision and puts the firm's operational decisions in a broader context. Further, we introduce the bank as a second player in the model in order to incorporate information asymmetry. Thus, the owner is not the only decision maker. This setting captures the impact of market conditions (e.g., the external market return, the bank's lending sentiment) on the equilibrium order quantity, the capital structure, and the risk of bankruptcy.

3 Model

We set up a single-period simultaneous game with two players, a small business owner and a commercial bank. Based on Buzacott and Zhang (2004) and Foley et al. (2012), the model seeks to capture the main features of the ABL industry.

The owner-newsvendor. The owner has capital K to invest and wishes to allocate this capital between a new business and the external capital market. Let α_m be a random variable denoting the rate of return on the external capital market. The new business is modeled as a single-period inventory (newsvendor) model, which is a standard way to represent capacity decisions under demand uncertainty. Let c denote the per unit procurement cost, s the salvage value, p the selling

price, and ξ the random demand for the newsvendor. The owner makes three related decisions: how much equity and debt to have in the newsvendor business and what stocking quantity to purchase. The first two constitute the capital structure of the firm, and the third its operational investment.

The owner constitutes the newsvendor business as a limited liability firm. This means that the owner and the firm are separate legal personalities, as per corporate law, and the owner's loss is limited to the amount of equity x in case the firm defaults on the loan. However, since the owner makes the stocking and borrowing decisions for the firm, their objectives are aligned. We use the terms newsvendor and firm interchangeably.

Let $x \in [0, K]$ denote the amount of equity of the newsvendor, w its borrowing amount, and q its stocking quantity. To formulate the business owner's problem, we write the ending cash position of the newsvendor as:

$$\begin{aligned}\pi(q, w, x, \xi) = \Big(& x + w - cq + p \min\{\xi, q\} + s(q - \xi)^+ - (1 + \alpha)w \\ & - \tau \left[p \min\{\xi, q\} + s(q - \xi)^+ - cq - \alpha w \right]^+ \Big)^+. \end{aligned} \tag{1}$$

Here, α denotes the interest rate charged by the bank on the loan, and τ denotes the corporate tax rate. Our assumption of tax payment is based on the capital structure literature, which deals with a tradeoff between the tax shield of debt and bankruptcy costs. Thus, the firm pays a tax if its pre-tax *operating income* is positive, and no tax if it incurs an operating loss. If the demand ξ is sufficiently large, the firm repays the loan plus interest, $(1 + \alpha)w$, to the bank and its after-tax *ending cash position* accrues to the owner. Otherwise, it files for bankruptcy liquidation and its cash and inventory are possessed by the bank. Note the effect of limited liability: the firm's cash position $\pi(q, w, x, \xi)$ is always non-negative because the risk of bankruptcy is shared with the bank.

We assume that the demand ξ is non-negative and follows a continuous probability distribution with increasing failure rate (IFR). The pdf, cdf, complementary cdf (ccdf), and inverse ccdf of the demand distribution are denoted as f, F, $\bar{F}$, and $\bar{F}^{-1}$, respectively, where f is positive on an

interval and zero elsewhere. Therefore, the owner solves the following problem:

$$\Pi^* \equiv \max_{(x,w,q)} \quad (1+\bar{\alpha}_m)(K-x) + E_F\left[\pi(q,w,x,\xi)\right] \tag{2}$$

$$\text{s.t.} \quad 0 \le cq \le x + w \tag{3}$$

$$0 \le w \le \psi(q) \tag{4}$$

$$0 \le x \le K. \tag{5}$$

Here, $\bar{\alpha}_m$ denotes the expected value of α_m, and E_F denotes expectation with respect to the distribution of demand. Constraint (3) specifies that the cost of procurement must be less than the total cash available (i.e., debt plus equity) to the firm. Constraint (4) limits the borrowing amount w to the credit limit set by the bank.

The tradeoff between investing in the external market and investing in the firm drives the owner's equity decision because the owner gives up the opportunity to earn $\bar{\alpha}_m x$ in the external market by investing x in the firm. Injecting more equity increases the firm's upside potential by allowing the firm to purchase more inventory, but also increases the maximum loss in case of a default, which equals x due to limited liability.

The bank. The bank is a monopoly and seeks to maximize its expected profit by lending to the firm. We assume that the bank knows the order quantity of the firm because it can access its balance sheet as well as audit the firm. However, it does not know the true demand distribution of the firm. Let $g, G, \bar{G}$ denote the pdf, cdf, and ccdf, respectively, of the bank's belief of the demand distribution. The bank uses this belief to assess the firm's loan application.

The bank charges a fixed interest rate α, and sets an asset-based credit limit to maximize its profit. Given the firm's inventory q, the bank decides a credit limit $\psi(q) = \gamma cq$, where γ is the collateral value of inventory. If the firm repays the loan with interest in full, then the bank makes a profit of αw. Otherwise, the bank liquidates the leftover inventory incurring a bankruptcy cost due to forced liquidation. We assume that the liquidation of the inventory happens at the salvage value s because the bank can access the same marketplace as the firm. We also assume that the bankruptcy cost is proportional to the size of bankruptcy, given by the amount of bank's loss or loan write-off in the event of bankruptcy.

Note that the optimal value of γ set by the bank depends on the tradeoff between the interest income on the loan and the expected write-off in case of default. As γ increases, the potential interest income increases, but the likelihood and size of bankruptcy also increase. Therefore, the bank solves a newsvendor-like problem.

Instead of a fixed interest rate, we could have allowed the bank to optimize the interest rate or set it to achieve a risk-free rate of return under its belief of the demand distribution. There are several reasons, both theoretical and practical, for modeling a fixed interest rate. This assumption mirrors the practical scenario of small businesses that obtain asset based loans under programs governed by the SBA. As we discussed in Section 1, the SBA sets a maximum interest rate and many small businesses borrow at that rate.[1] Theoretically, a fixed interest rate with a credit limit is justified due to the adverse selection problem faced by the bank under information asymmetry. Increasing the interest rate can lower the bank's profit by changing the set of borrowing firms and inducing firms to make riskier investments (Stiglitz and Weiss 1981). The literature on adverse selection implies that the lending interest rate should be optimized for a population of borrowers, not for an individual firm; see Greenwald and Stiglitz (1990; page 15). It also implies that a credit allocation mechanism, such as asset based lending, mitigates adverse selection by preventing over-borrowing. In Appendix A, we discuss these issues in detail, present alternative models with interest rate optimization, and illustrate that using ABL leads to a higher expected profit for the bank than optimizing the interest rate without a credit limit. Finally, a fixed interest rate helps simplify our model and focus on the implications of asset based lending.

Liquidation of leftover inventory is an important basis for ABL. Practically, if inventory had no salvage value, then the loan would be unsecured. Its interest rate would be higher and it would bear

[1] Bates (2000, page 230) discusses examples of small firms that borrow at the maximum interest rate set by the SBA, e.g., "Exim Capital [an asset based lender], [is] typically charging its borrowers interest rates in the 15% annual range... Chun [a manager at Exim Capital] would like to charge 18%, but SBA regulations hold him currently to 15%...[T]he collateral [is] involved in six typical loans made by Exim Capital, which involved secured loans to (a) Jee and Jung Cleaners, (b) C.H.A. Kyung, Inc., (c) Tri J. Tires, (d) H.S.K. Cleaners, (e) 104 Broadway Farm (a grocery), and (f) Chonel, Inc. Loan sizes ranged from \$52,000 to \$105,000 in these six transactions...[T]he six deals offered collateral to Exim Capital ranging from \$162,500 to \$801,000." Lending interest rates are also closely monitored and capped in developing countries with a history of loan-sharking (Park 1973).

stricter financial covenants and monitoring. Mathematically, we will show later that the salvage value yields a lower bound on the credit limit because it is the minimum amount for which the inventory can be disposed off. A good illustration of the usefulness of salvaging inventory in ABL is provided by the bankruptcy filing and the subsequent inventory liquidation of the Borders Group, Inc. Borders obtained an asset based loan of $700 million in April 2010 from a consortium of lenders (ABF Journal 04/01/2010). After its Chapter 11 bankruptcy protection in February 2011 and Chapter 7 liquidation in July 2011, Gordon Brothers Group and Hilco Merchant Resources sold Borders inventories at 40-60 percent discounts (Legal News 09/05/2011).

We do not apply corporate taxes to the bank's income or personal taxes to the owner because there are no economic tradeoffs related to these taxes in our model. Taxes will merely scale the respective incomes without changing the nature of results. Therefore, we ignore them for simplicity, and assume that the bank and the owner live for more than one period and have other sources of income that they can offset losses against.

All parameters other than the newsvendor's true demand distribution are common knowledge. For the newsvendor problem to be non-trivial, we assume that $(1-\tau)(p-s) > (1+\alpha)c - s$ and $c > s$. We also require that $\frac{p-s}{(1+\alpha)c-s} \geq 1+\alpha$, i.e., the profit margin of the newsvendor is sufficiently high so that the rate of return of a sold unit that is purchased on credit, $\frac{p-s}{(1+\alpha)c-s} - 1$, is no less than the borrowing rate α. These assumptions are necessary to ensure that borrowing is a profitable option for the firm.

We first derive the asset based credit limit. Then, we solve the owner's optimization problem given the bank's credit limit decision to obtain the equilibrium outcome. While we model the interaction of the bank and the firm as a simultaneous game, in practice, the events will occur over a period of time. For example, the owner obtains the inventory q on credit from the supplier, then gets the loan from the bank by securing the inventory, and subsequently pays the supplier. This practice is described by Foley et al. (2012). It boils down to the simultaneous game described above.

3.1 The Bank's Problem

We first note that the firm will never borrow and hold cash because of the interest levied on the loan. Therefore,

$$w = (cq - x)^+. \tag{6}$$

Thus, the bank's problem collapses into two variables, q and w, because x is fully determined whenever the credit limit is binding. Let $\kappa(q, w, \xi)$ denote the bank's profit as a function of the inventory level q, the loan amount w, and demand occurrence ξ. Given q and w, there is a threshold value of demand below which the firm is unable to repay the loan and interest in full and bankruptcy occurs. Let d_B denote this threshold. Observe from (1) that when the firm is bankrupt, its operating income cannot be positive and there are zero taxes. Using this and setting $\pi(q, w, x, d_B)$ equal to zero gives the bankruptcy threshold $d_B = \left(\frac{1+\alpha}{p-s}w - \frac{s}{p-s}q\right)^+$ as a function of w and q. This threshold implies that the firm survives with probability 1 if it does not borrow or if the borrowing amount $w \leq \frac{s}{1+\alpha}q$. If $w > \frac{s}{1+\alpha}q$, then there are two scenarios. If the realized demand exceeds d_B, then the firm survives and repays the loan plus interest to the bank. In contrast, if the realized demand is less than d_B, then the bank receives a cash amount of $p\xi$ and an unsold inventory of $q - \xi$ from the firm. Hence, if $\xi < d_B$, then the bank's profit after forced liquidation is

$$\kappa\left(q, w, \xi\right) = p\xi + s(q - \xi) - b(d_B - \xi) - w. \tag{7}$$

Here, $b \geq 0$ is the bankruptcy cost per unit and the size of the bankruptcy is given by the difference between the bankruptcy threshold demand d_B and the realized demand ξ. We rewrite (7) by using the definition of d_B and by noting that $w = cq - x$ when $d_B > 0$. Thus, combining all the scenarios together, we find that the bank's profit is given by:

$$\kappa\left(q, w, \xi\right) = \alpha w - (p - s + b)\left[d_B - \xi\right]^+. \tag{8}$$

This expression is equal to zero if the firm does not borrow, equal to αw if the firm borrows and survives, and equal to (7) if the firm borrows and defaults. It decomposes into two terms, the first gives the profit when there is no bankruptcy, and the second captures the amount of loan write-off in case the newsvendor defaults on its loan.

Since the asset based credit limit is expressed as $\psi(q) = \gamma cq$, the bank's objective is to choose $\gamma \in [0, 1]$ for the newsvendor's loan application in order to maximize $E_G[\kappa(q, \gamma cq, \xi)]$ under its belief G, i.e.,

$$\max_{0 \le \gamma \le 1} \quad \alpha\gamma cq - (p - s + b) \int_0^{\left(\frac{1+\alpha}{p-s}\gamma cq - \frac{s}{p-s}q\right)^+} G(\xi)d\xi. \tag{9}$$

Here, $\gamma \le 1$ in order to ensure the non-negativity of the owner's equity investment. In the following proposition, we derive the optimal inventory collateral value and the corresponding asset based credit limit. All proofs are presented in Appendix B.

Proposition 1 *Let* $\theta = \frac{p-s}{(1+\alpha)c-s}\bar{G}^{-1}\left(1 - \left(\frac{\alpha}{1+\alpha}\right)\frac{p-s}{p-s+b}\right)$. *The optimal collateral value of inventory is*

$$\gamma^*(q) = \begin{cases} 1 & \text{if } q \le \theta, \\ \frac{s}{(1+\alpha)c} + \left(1 - \frac{s}{(1+\alpha)c}\right)\frac{\theta}{q} & \text{if } q > \theta, \end{cases} \tag{10}$$

and the corresponding asset based credit limit offered by the bank is equal to $\psi^*(q) = \gamma^*(q)cq$.

Proposition 1 shows the role of the salvage value of inventory in ABL. When $s = 0$, the credit limit for large values of q equals $c\theta$, which is independent of q. Hence, when $s = 0$, $\psi^*(q)$ can be interpreted as an unsecured credit line.

We illustrate the optimal collateral value of inventory and the corresponding credit limit as functions of q in Figure 1. Note that when q is sufficiently small, $\gamma^*(q) = 1$, i.e., the bank is willing to finance the entire inventory procurement cost of cq. As the firm's inventory increases, the bank lowers the collateral value because the probability of having unsold units, which will be salvaged at a discount, increases in q. Thus, $\gamma^*(q)$ approaches the relative salvage value $\frac{s}{(1+\alpha)c}$ as q increases. Moreover, the portion of inventory that is financed through a bank loan declines as q increases. We describe further managerial insights regarding the credit limit in Section 4.3 after solving for the equilibrium.

3.2 The Business Owner's Problem

In this section, we solve the business owner's problem given the asset based credit limit $\psi^*(q)$ in three steps. We first find the optimal order quantity $q^*(x)$ for a given equity investment x.

Then we find the equilibrium equity investment x^* that maximizes the owner's expected ending cash position. Finally, we show that this sequential optimization procedure, which helps us avoid complicated first and second order partial derivatives, gives the same expected ending position as joint optimization over x and q.

3.2.1 The Stocking Decision

Following (6), we can eliminate w and rewrite (3) and (4) as a single constraint. Thus, for given equity x, the newsvendor solves

$$\pi^*(x) \equiv \max_q \quad E_F\left[\pi(q, (cq-x)^+, x, \xi)\right] \quad \text{s.t.} \quad 0 \le cq \le x + \psi^*(q). \tag{11}$$

We rewrite the credit constraint in (11) by determining feasible (x, q) pairs that ensure that the asset based credit limit imposed by the bank is not violated and that $x + w$ is sufficient to pay the procurement cost. If $q \le \theta$, then Proposition 1 implies that $\psi^*(q) = cq$. Hence, the constraint becomes $x \ge 0$, which implies that (x, q) pairs with $x \ge 0$ and $q \le \theta$ are feasible. If $q > \theta$, then Proposition 1 implies that $\psi^*(q) = \frac{s}{1+\alpha}q + \frac{(1+\alpha)c-s}{1+\alpha}\theta$. Hence, the constraint in (11) implies that $x \ge \frac{(1+\alpha)c-s}{1+\alpha}(q-\theta) \ge 0$. Thus, $q \le \beta x + \theta$ must hold for a given x, where $\beta \equiv \frac{1+\alpha}{(1+\alpha)c-s}$. The union of two feasible subsets with $q \le \theta$ and $q > \theta$ implies that (x, q) pairs with $x \ge 0$ and $q \le \beta x + \theta$ are feasible. On the contrary, q values that exceed $\beta x + \theta$ are infeasible because the credit limit offered by the bank for such q values is insufficient to pay the supplier.

The above steps allows us to rewrite the newsvendor's problem as a function of non-negative starting equity x as

$$\pi^*(x) = \max_{q \ge 0} \quad E_F\left[\pi(q, (cq-x)^+, x, \xi)\right] \quad \text{s.t.} \quad q \le \beta x + \theta. \tag{12}$$

Three possible scenarios can occur based on the values of the starting capital and the order quantity of the newsvendor. We denote as (NB) the scenario in which the newsvendor has enough cash to purchase inventory and does not borrow any money from the bank. In scenario (BWO), the newsvendor's borrowing amount is sufficiently small as to be without bankruptcy risk. In scenario (BWR), the newsvendor borrows with bankruptcy risk. We use the superscripts NB, BWO and BWR to denote variables in the respective solutions. Figure 2 depicts the regions defining these scenarios as functions of the order quantity q, the demand realization ξ, and the equity x.

We now characterize the newsvendor's expected ending cash position in each scenario. Scenario (NB) occurs when $q \leq x/c$. In this scenario, the newsvendor has operating income and pays tax if $\xi \geq \frac{c-s}{p-s}q$. It has operating loss if $\xi < \frac{c-s}{p-s}q$. Thus, its ending cash position is

$$\pi^{NB}(q,0,x,\xi) = x + p\min\{\xi,q\} + s(q-\xi)^+ - cq - \tau(p-s)\left[\min\{\xi,q\} - \frac{c-s}{p-s}q\right]^+.$$

By taking an expectation, we obtain the newsvendor's expected ending cash position in (NB) as

$$E_F\left[\pi^{NB}\right] = x + (p-s)\left(\int_0^q \bar{F}(\xi)d\xi - \tau\int_{\frac{c-s}{p-s}q}^q \bar{F}(\xi)d\xi\right) - (c-s)q.$$

Scenario (BWO) occurs when the order quantity is greater than x/c, but is sufficiently small so that the newsvendor does not default even when realized demand is zero. For $q > x/c$, the newsvendor's ending cash position when it realizes demand $\xi = 0$ is given by:

$$(1+\alpha)x - [(1+\alpha)c - s]q - \tau\big(\alpha x - [(1+\alpha)c - s]q\big)^+. \tag{13}$$

Observe that $(\alpha x - [(1+\alpha)c - s]q)^+ = 0$ for any $q > x/c$. That is, the newsvendor has operating loss, which implies that q must not exceed βx to ensure that (13) is non-negative. Thus, in scenario (BWO), $x/c < q \leq \beta x$. Further, to compute the newsvendor's ending cash position, we note that the newsvendor has taxable income if $\xi \geq d_T(q,x) \equiv \frac{(1+\alpha)c-s}{p-s}q - \frac{\alpha}{p-s}x$, and loss otherwise. Therefore, the newsvendor's ending cash position in (BWO) is

$$\pi^{BWO}(q,w,x,\xi) = (1+\alpha)(x-cq) + p\min\{\xi,q\} + s(q-\xi)^+ - \tau(p-s)\left[\min\{\xi,q\} - d_T(q,x)\right]^+.$$

Taking an expectation gives

$$E_F\left[\pi^{BWO}\right] = (1+\alpha)x + (p-s)\left(\int_0^q \bar{F}(\xi)d\xi - \tau\int_{d_T(q,x)}^q \bar{F}(\xi)d\xi\right) - [(1+\alpha)c - s]q.$$

Scenario (BWR) occurs and the newsvendor borrows with bankruptcy risk if $q > \beta x$. For each such value of q, recall from the previous subsection that the firm is bankrupt if the realized demand is below $d_B(q,x) = \left(\frac{1+\alpha}{p-s}w - \frac{s}{p-s}q\right)^+ = \left(\frac{(1+\alpha)c-s}{p-s}q - \frac{1+\alpha}{p-s}x\right)^+$. For $d_B(q,x) \leq \xi \leq d_T(q,x)$, the newsvendor survives, but has an operating loss. The newsvendor earns an operating income if

demand is higher than the threshold $d_T(q,x)$. Thus, we get

$$\pi^{BWR}(q,w,x,\xi) = \begin{cases} 0 & \text{if } \xi \le d_B(q,x), \\ (p-s)\left[\min\{\xi,q\} - d_B(q,x)\right] & \text{if } d_B(q,x) < \xi \le d_T(q,x), \\ (p-s)\left[d_T(q,x) - d_B(q,x)\right] & \\ \quad +(1-\tau)(p-s)\left[\min\{\xi,q\} - d_T(q,x)\right] & \text{if } \xi > d_T(q,x). \end{cases}$$

Taking an expectation gives

$$E_F\left[\pi^{BWR}\right] = (p-s)\left[\int_{d_B(q,x)}^{q} \bar{F}(\xi)d\xi - \tau\int_{d_T(q,x)}^{q} \bar{F}(\xi)d\xi\right].$$

The above analysis defines the cutoff values of inventory and demand for the three scenarios and specifies the expected ending cash position of the newsvendor for each scenario. Note that the cutoff values of demand are d_T and d_B, which respectively denote the minimum demand above which the newsvendor makes a profit and pays tax, and the minimum demand above which the newsvendor does not go bankrupt. Both cutoffs are functions of equity and inventory. We shall drop the arguments of these functions for notational convenience. For instance, d_B denotes $d_B(q,x)$. Let $\tilde{q}(x)$ denote the optimal order quantity for the newsvendor as a function of its starting capital when there is no credit limit. Solving the newsvendor's problem in the three scenarios, we show that $\tilde{q}(x)$ has the following form.

Proposition 2 *Let $q^{BWR}(x), q^{BWO}(x), q^{NB}$ be order quantities defined by:*

$$q^{BWR}(x) = \bar{F}^{-1}\left(\frac{(1+\alpha)c-s}{(1-\tau)(p-s)}\left[\bar{F}\left(d_B\right) - \tau\bar{F}\left(d_T\right)\right]\right), \tag{14}$$

$$q^{BWO}(x) = \bar{F}^{-1}\left(\frac{(1+\alpha)c-s}{(1-\tau)(p-s)}\left[1 - \tau\bar{F}\left(d_T\right)\right]\right), \tag{15}$$

$$q^{NB} = \bar{F}^{-1}\left(\frac{c-s}{(1-\tau)(p-s)}\left[1 - \tau\bar{F}\left(\frac{c-s}{p-s}q^{NB}\right)\right]\right). \tag{16}$$

The optimal order quantity for the newsvendor without the quantity constraint, $\tilde{q}(x)$, is given by $q^{BWR}(x)$ if $0 \le x < x_2$, $q^{BWO}(x)$ if $x_2 \le x < x_3$, x/c if $x_3 \le x \le x_4$, q^{NB} if $x > x_4$, where $x_2 \le x_3 \le x_4$ are cutoff values of the newsvendor's equity. These cutoff values are uniquely defined by $x_2 = \frac{(1+\alpha)c-s}{1+\alpha}q^{BWR}(x_2)$, $x_3 = cq^{BWO}(x_3)$, $x_4 = cq^{NB}$.

Intuitively, the newsvendor borrows with bankruptcy risk if its equity is less than x_2, borrows without bankruptcy risk if its equity lies between x_2 and x_3, and does not borrow if its equity is

greater than x_3. In the last case, $x > x_4$ and the optimal order quantity does not vary with x because the cash constraint is not binding.

In the next proposition, we show that the optimal order quantity is a non-monotone function of equity.

Proposition 3 *$\tilde{q}(x)$ is continuous in x. It decreases in x when x is sufficiently small, increases in x for $x_2 \leq x \leq x_4$ and is constant for $x \geq x_4$.*

The main inference from this proposition is that the optimal order quantity is decreasing in equity for small values of x. This property will be useful in solving for the equilibrium order quantity. Intuitively, it means that the investor's tendency to take riskier operational decisions increases when she injects less equity. This occurs due to the existence of a moral hazard problem under limited liability. See Easterbrook and Fischel (1985) for a general discussion on the incentives created by limited liability to transfer risk to debt holders.

Following (12) and Proposition 2, the order quantity at equilibrium is given by $\tilde{q}(x)$ if the credit limit is not binding, and by $q^L(x) \equiv \beta x + \theta$ otherwise. $q^L(x)$ can be interpreted as the optimal order quantity from the bank's perspective since it maximizes the bank's expected profit under G. Lemma 1 proves a necessary and sufficient condition for these two functions to intersect, and show when the credit limit is and is not binding.

Lemma 1 *If $q^{BWR}(0) \leq q^L(0)$ then the credit limit is never binding. If $q^{BWR}(0) > q^L(0)$, then there exists an equity value x_1 such that $0 < x_1 < x_2$ and $q^{BWR}(x_1) = q^L(x_1)$ and the credit limit is binding for $x \in [0, x_1]$. The value of x_1 is given by*

$$(1-\tau)(p-s)\bar{F}(\beta x_1+\theta)+\tau[(1+\alpha)c-s]\bar{F}\left(\frac{x_1+[(1+\alpha)c-s]\theta}{p-s}\right) = [(1+\alpha)c-s]\bar{F}\left(\frac{(1+\alpha)c-s}{p-s}\theta\right). \tag{17}$$

Figure 3 illustrates the outcome of the newsvendor-bank interaction. In Figure 3(a), the amount that the bank is willing to lend to the firm with zero equity is less than the amount that the firm wants to borrow with zero equity. Thus, x_1 exists. The newsvendor's order quantity is restricted by the credit limit when $x \leq x_1$ and unrestricted otherwise. In Figure 3(b), $q^{BWR}(0) \leq q^L(0)$, so that the newsvendor's order quantity is nowhere restricted by the credit limit. The fact that

$q^{BWR}(0)$ can be less than $q^L(0)$ is interesting because it shows that the credit line offered to an all-debt newsvendor can exceed its unconstrained optimal borrowing amount.

We can now specify the equilibrium solution. When $q^{BWR}(0) > q^L(0)$, the newsvendor's order quantity is given by:

$$q^*(x) = \begin{cases} q^L(x) & \text{if } 0 \le x \le x_1 \text{ (Case 1)}, \\ q^{BWR}(x) & \text{if } x_1 < x < x_2 \text{ (Case 2)}, \\ q^{BWO}(x) & \text{if } x_2 \le x < x_3 \text{ (Case 3)}, \\ x/c & \text{if } x_3 \le x \le x_4 \text{ (Case 4)}, \\ q^{NB} & \text{if } x > x_4 \text{ (Case 5)}. \end{cases} \tag{18}$$

Correspondingly, the newsvendor's expected ending cash position is

$$\pi^*(x) = \begin{cases} (p-s)\left[\int_{d_B}^{q^L} \bar{F}(\xi)d\xi - \tau \int_{d_T}^{q^L} \bar{F}(\xi)d\xi\right] & \text{if } 0 \le x \le x_1, \\ (p-s)\left[\int_{d_B}^{q^{BWR}} \bar{F}(\xi)d\xi - \tau \int_{d_T}^{q^{BWR}} \bar{F}(\xi)d\xi\right] & \text{if } x_1 < x < x_2, \\ (1+\alpha)x - ((1+\alpha)c - s)q^{BWO} + (p-s)\left[\int_0^{q^{BWO}} \bar{F}(\xi)d\xi - \tau \int_{d_T}^{q^{BWO}} \bar{F}(\xi)d\xi\right] & \text{if } x_2 \le x < x_3, \\ \frac{s}{c}x + (p-s)\left[\int_0^{x/c} \bar{F}(\xi)d\xi - \tau \int_{\frac{(c-s)x}{(p-s)c}}^{x/c} \bar{F}(\xi)d\xi\right] & \text{if } x_3 \le x \le x_4, \\ x - (c-s)q^{NB} + (p-s)\left[\int_0^{q^{NB}} \bar{F}(\xi)d\xi - \tau \int_{\frac{c-s}{p-s}q^{NB}}^{q^{NB}} \bar{F}(\xi)d\xi\right] & \text{if } x > x_4. \end{cases} \tag{19}$$

Note that $q^*(x)$ and $\pi^*(x)$ are defined in terms of five cases. In Case 1, the newsvendor borrows with bankruptcy risk and the bank's credit limit is binding. In Case 2, the newsvendor borrows with risk, but the bank's credit limit is not binding. Cases 1 and 2 are subsumed in scenario (BWR) defined earlier. In Case 3, the newsvendor borrows without bankruptcy risk; this corresponds to scenario (BWO). In Case 4, the newsvendor does not borrow and uses up all the equity to procure inventory. In Case 5, the newsvendor does not borrow, is left with excess cash. Cases 4 and 5 correspond to scenario (NB). When $q^{BWR}(0) \le q^L(0)$, the equilibrium solution is similar except that Case 1 does not arise.

In the next section, we will see that some of the five cases do not occur at equilibrium once the owner's equity investment problem is introduced. We present the solution to the owner's problem, first under the condition $q^{BWR}(0) > q^L(0)$, then under $q^{BWR}(0) \le q^L(0)$.

3.2.2 The Equity Investment Decision

Having determined $q^*(x)$, we now solve a capital allocation problem to determine the amount x of equity of the newsvendor and $K - x$ of investment in an external market asset. The owner's problem is formulated as:

$$\Pi^{S^*} \equiv \max_{x\in[0,K]} \Pi^S(x) = \max_{x\in[0,K]} (1+\bar{\alpha}_m)(K-x) + \pi^*(x),$$

where the superscript S denotes sequential optimization. $\pi^*(x)$ is given by (19). We assume that K is sufficiently large to procure the optimal order quantity of a pure equity newsvendor, q^{NB}. This assumption is made only to ease the presentation because it guarantees the feasibility of Case 5 in (18).

We solve the owner's problem by determining the optimal solution in each of the Cases 1-5 and then finding the highest value. We first show that Case 2 (i.e., $x \in (x_1, x_2)$) cannot arise in equilibrium.

Lemma 2 *$\Pi^S(x)$ is convex in x for $x \in (x_1, x_2)$. Thus, borrowing with risk but ordering less than the bank's optimal order quantity q^L cannot arise in equilibrium.*

As a consequence of this lemma, if the firm borrows with bankruptcy risk then the credit limit is binding. Thus, the optimal equity investment with bankruptcy risk lies in the range $x \in [0, x_1]$. Its value is given by the following proposition.

Proposition 4 *Let $\alpha_l = \alpha\frac{(1-\tau)(p-s)}{(1+\alpha)c-s}\bar{F}(\beta x_1+\theta)+\bar{F}\left(\frac{(1+\alpha)c-s}{p-s}\theta\right)-1$ and $\alpha_h = (1+\alpha)\frac{(1-\tau)(p-s)}{(1+\alpha)c-s}\bar{F}(\theta)+\tau\bar{F}\left(\frac{(1+\alpha)c-s}{p-s}\theta\right)-1$. The optimal equity investment with bankruptcy risk, x_R^*, is given by*

$$x_R^* = \begin{cases} x_1 & \text{if } \bar{\alpha}_m < \alpha_l, \\ \tilde{x}_R & \text{if } \bar{\alpha}_m \in [\alpha_l, \alpha_h], \\ 0 & \text{if } \bar{\alpha}_m > \alpha_h, \end{cases}$$

where $\tilde{x}_R$ solves

$$(1-\tau)\frac{(1+\alpha)(p-s)}{(1+\alpha)c-s}\bar{F}(\beta\tilde{x}_R+\theta) + \tau\bar{F}\left(\frac{\tilde{x}_R + [(1+\alpha)c-s]\theta}{p-s}\right) - 1 = \bar{\alpha}_m. \tag{20}$$

Intuitively, when $\bar{\alpha}_m$ is relatively high (i.e., $\bar{\alpha}_m > \alpha_h$), the owner does not invest any amount in the newsvendor because her opportunity cost is high. When $\bar{\alpha}_m$ is relatively low (i.e., $\bar{\alpha}_m < \alpha_l$), the owner invests as much as she can (i.e., $x = x_1$). For intermediate values of $\bar{\alpha}_m$, the owner chooses a value of equity in order to match the marginal return from the newsvendor, given by the left hand side of (20), with $\bar{\alpha}_m$.

Now consider the cases when the newsvendor does not face any risk of bankruptcy, i.e., when $x \in [x_2, K]$. The optimal equity investment value without bankruptcy risk, x^*_{NR} is given by the following proposition.

Proposition 5 *Let* $\alpha_2 = \alpha\left[1 - \tau\bar{F}\left(\frac{x_2}{p-s}\right)\right]$ *and* $\alpha_3 = \alpha\left[1 - \tau\bar{F}\left(\frac{c-s}{p-s}q^{BWO}(x_3)\right)\right]$. *The optimal equity investment in* $[x_2, K]$, *i.e., without bankruptcy risk, is*

$$x^*_{NR} = \begin{cases} x_2 & \text{if } \bar{\alpha}_m > \alpha_2, \\ x^*_3 & \text{if } \bar{\alpha}_m \in (\alpha_3, \alpha_2], \\ x^*_4 & \text{if } \bar{\alpha}_m \in [0, \alpha_3]. \end{cases}$$

where x^*_3 *solves*

$$\bar{F}\left(q^{BWO}(x^*_3)\right) = \frac{\bar{\alpha}_m[(1+\alpha)c - s]}{\alpha(1-\tau)(p-s)} \quad (\textit{Case 3}),$$

and x^*_4 *solves*

$$(1-\tau)(p-s)\bar{F}\left(\frac{x^*_4}{c}\right) + \tau(c-s)\bar{F}\left(\frac{c-s}{p-s}\left[\frac{x^*_4}{c}\right]\right) = (1+\bar{\alpha}_m)c - s \quad (\textit{Case 4}).$$

Here x^*_3 corresponds to the optimal equity investment value if the firm borrows without risk (i.e., if the optimal equity investment is in $(x_2, x_3]$), and x^*_4 corresponds to the optimal equity investment value if the firm does not borrow (i.e., if the optimal equity investment is in $(x_3, x_4]$). To derive the solution in Proposition 5, we show that the owner's objective function is concave in Cases 3-5, and the first derivatives from the right and left are equal to each other at every switching point. Therefore, there is a unique local optimum, and collecting the three cases together gives the optimal solution under no borrowing. The owner's solution lies at the left boundary x_2 when $\bar{\alpha}_m > \alpha_2$ because the external asset offers a very attractive investment alternative. As the external investment alternative becomes less promising (i.e., when $\bar{\alpha}_m \in (\alpha_3, \alpha_2]$), the owner uses a mix of riskless debt and equity to finance her firm's operations. For smaller $\bar{\alpha}_m$ values (i.e.,

for $\bar{\alpha}_m \in [0, \alpha_3]$), the owner creates a pure equity firm by investing x_4^*, which sets the after-tax marginal return from the newsvendor equal to $\bar{\alpha}_m$.

Let $\Pi_R^{S^*}(\bar{\alpha}_m)$ and $\Pi_{NR}^{S^*}(\bar{\alpha}_m)$ denote the owner's optimal payoff functions with and without bankruptcy risk under sequential optimization, respectively. We find that $\Pi_R^{S^*}$ and $\Pi_{NR}^{S^*}$ are both increasing in $\bar{\alpha}_m$, but $\Pi_R^{S^*}$ is increasing at a faster rate. Moreover, $\Pi_R^{S^*} > \Pi_{NR}^{S^*}$ when $\bar{\alpha}_m$ is sufficiently large. Therefore, if $\Pi_R^{S^*}(0) > \Pi_{NR}^{S^*}(0)$ at $\bar{\alpha}_m = 0$, then the two functions never intersect and investing with risk is optimal for all $\bar{\alpha}_m \geq 0$. This scenario can arise if the bank offers a sufficiently attractive borrowing opportunity with a high credit limit and/or a low interest rate. On the other hand, if $\Pi_R^{S^*}(0) \leq \Pi_{NR}^{S^*}(0)$, then the two functions intersect at a unique threshold value of $\bar{\alpha}_m$. Let $\tilde{\alpha}$ denote this threshold. If $\bar{\alpha}_m \leq \tilde{\alpha}$, then the owner invests enough equity into the newsvendor that the probability of bankruptcy is zero. Otherwise, the owner finds it optimal to invest with bankruptcy risk. Proposition 6 formalizes this result.

Proposition 6 *If $\Pi_R^{S^*}(0) > \Pi_{NR}^{S^*}(0)$ then investing with bankruptcy risk is optimal for all $\bar{\alpha}_m \geq 0$. Otherwise, there exists a unique threshold return value $\tilde{\alpha} \geq 0$ such that investing without bankruptcy risk is optimal when $\bar{\alpha}_m \leq \tilde{\alpha}$ and investing with bankruptcy risk is optimal otherwise.*

Finally, we show that sequential optimization leads to the same optimal solution for the owner as the joint optimization.

Proposition 7 *Sequential optimization over $q(x)$ and x leads to the same expected ending cash position for the owner as joint optimization over q and x. That is, $\Pi^* = \Pi^{S^*}$.*

With reference to Lemma 1, we have thus far demonstrated the interaction between the owner and the bank when x_1 exists, i.e., when the credit limit can be binding for some range of equity investments. It may be noted that the above solution continues to hold even if the credit limit is never binding, i.e., in the scenario shown in Figure 3(b). When $q^L(0) \geq q^{BWR}(0)$, the bank's credit limit is greater than the newsvendor's optimal purchase quantity for $x \in (0, x_2)$. Thus, Case 1, i.e., borrowing with risk and ordering the bank's optimal quantity, does not arise because the credit limit is never binding. Further, from Lemma 2, the owner's payoff function is convex in Case 2. Therefore, the optimal solution for the borrowing with risk scenarios is either at $x = 0$ or $x = x_2$.

The optimal solution for borrowing without bankruptcy risk, Proposition 5, remains unchanged because Cases 3-5 are unaffected by the credit limit. Combining these together, the owner's global optimal solution when $q^L(0) \geq q^{BWR}(0)$ is at either $x^* = 0$ or $x^* = x^*_{NR}$. Writing the owner's payoff functions for $x = 0$ and $x = x^*_{NR}$ and comparing them, we again obtain a threshold value such that $x = 0$ is optimal for $\bar{\alpha}_m$ values that exceed the threshold and $x = x^*_{NR}$ is optimal for $\bar{\alpha}_m$ values that are below the threshold. We omit this step because it is analogous to Proposition 6.

In general, the value of $\tilde{\alpha}$ can be determined by a numerical search technique to find the intersection point between $\Pi^{S^*}_R(\bar{\alpha}_m)$ and $\Pi^{S^*}_{NR}(\bar{\alpha}_m)$.

4 Managerial Implications

We first present an example with zero taxes to illustrate the roles of different features of our model in the results. Then we discuss specific implications arising from our analysis.

The solution under zero taxes, shown in Table 1, can be a pure equity firm, one with both debt and equity, or a pure debt firm. The equilibrium order quantity equals $\tilde{q}_{NR}$, $\tilde{q}_R$ or θ in the three cases, respectively. Which of the three cases arises at equilibrium depends on market conditions through $\bar{\alpha}_m, \alpha$ and the bank's belief. Observe that the overage and underage costs of the classical newsvendor model are adjusted to capture market conditions. For example, $\bar{\alpha}_m$ can be interpreted as the cost of capital of a pure equity firm, and is incorporated in $\tilde{q}_{NR}$ because the cost of purchasing one unit equals $(1 + \bar{\alpha}_m)c$. The borrowing interest α is additionally incorporated in the formulas for $\tilde{q}_R$ and θ. As per Proposition 6, any of the three cases in Table 1 can occur when $\tilde{\alpha}$ exits, but the pure equity case does not occur otherwise.

One implication of this example is that a debt-equity mix or a pure debt firm can occur at equilibrium even in the absence of taxes. This result differs from the tradeoff theory originating from Kraus and Litzenberger (1973) because the objective function of the business owner in our model is to maximize her expected profit rather than the sum of the values of debt and equity. Thus, if the lending terms offered by the bank (i.e., the interest rate and the credit limit) are favorable then the firm might borrow even in the absence of taxation because borrowing increases the expected return to the owner and shifts the risk of bankruptcy to the bank. Conversely, if the

Table 1: Possible equilibrium values of inventory, debt, and equity in the absence of taxation when $\tilde{\alpha}$ exits (i.e., when $\Pi_R^{S^*}$ and $\Pi_{NR}^{S^*}$ intersect). If $\Pi_R^{S^*}$ and $\Pi_{NR}^{S^*}$ never intersect then the equilibrium outcome is debt equity mix for $0 \leq \bar{\alpha}_m \leq \alpha_h$ and pure debt for $\bar{\alpha}_m > \alpha_h$. The values of β and θ are defined in Proposition 1. $\tilde{q}_{NR}$ and $\tilde{q}_R$ can be obtained by setting $\tau = 0$ in Propositions 4 and 5. Borrowing without risk cannot arise as an equilibrium outcome when $\tau = 0$.

Possible Case	Order Quantity (q^*)	Equity (x^*)	Debt (w^*)
Pure equity ($0 \leq \bar{\alpha}_m \leq \tilde{\alpha}$)	$\tilde{q}_{NR} = \bar{F}^{-1}\left(\frac{(1+\bar{\alpha}_m)c-s}{p-s}\right)$	$c\tilde{q}_{NR}$	0
Debt equity mix ($\tilde{\alpha} < \bar{\alpha}_m \leq \alpha_h$)	$\tilde{q}_R = \bar{F}^{-1}\left(\frac{(1+\bar{\alpha}_m)\left(c-\frac{s}{1+\alpha}\right)}{p-s}\right)$	$\frac{\tilde{q}_R-\theta}{\beta}$	$\frac{s}{1+\alpha}\tilde{q}_R + \frac{\theta}{\beta}$
Pure debt ($\bar{\alpha}_m > \alpha_h$)	θ	0	$c\theta$

lending terms are not favorable then the owner might choose to use pure equity even in the absence of bankruptcy costs. (In our model, bankruptcy costs are embedded in θ, and our analysis remains valid when $b = 0$.) Thus, the optimal capital structure arises due to the tradeoff between the expected return from the newsvendor and that in the external market. In contrast, in the tradeoff theory, the optimal capital structure balances a tradeoff between tax benefits of debt and cost of bankruptcy.

Another implication is regarding the positive inventory liquidation value. Setting $s = 0$ in Table 1, we see that $\beta = \frac{1+\alpha}{(1+\alpha)c-s} = \frac{1}{c}$ and the credit limit reduces to $c\theta$, which is independent of the amount of inventory procured by the firm. Thus, a positive salvage value is essential for the credit constraint in the firm's optimization problem to depend on its order quantity. When $s = 0$, the firm can first find the optimal order quantity, then decide how to finance its procurement because the ordering decision does not change its borrowing ability. As a consequence, the potential equilibrium order quantities under pure equity and debt-equity mix are equal, i.e., $\tilde{q}_R = \tilde{q}_{NR} = \bar{F}^{-1}\left(\frac{(1+\bar{\alpha}_m)c}{p}\right)$.

4.1 Information Asymmetry

The solution shown in Table 1 incorporates the bank's beliefs through θ and the threshold $\tilde{\alpha}$. The bank's view of the newsvendor's demand may be optimistic or pessimistic compared to the true distribution. If F has first order stochastic dominance over G, then the bank is pessimistic, otherwise it is optimistic. As the bank gets more pessimistic, it tightens the credit limit by changing

θ as shown in the inventory collateral formula (10). Credit tightening decreases the firm's leverage as expected. If this were the only consideration, we would expect the order quantity to be decreasing in the bank's degree of pessimism. However, the degree of pessimism also affects the owner's equity investment decision through $\tilde{\alpha}$ and the credit limit. As a result, the optimal order quantity need not always decrease as the bank gets more pessimistic.

Table 2 illustrates this outcome by showing the equilibrium order quantity in two scenarios, (i) when the bank is optimistic and (ii) when the bank is pessimistic. We vary $\bar{\alpha}_m$ and keep all other parameters identical between the two scenarios. In practice, $\bar{\alpha}_m$ may be correlated with other model parameters. Since $\tilde{\alpha}$ varies with the degree of optimism, we find that $\tilde{\alpha}$ equals 9.68% and 11.85% in the games with the optimistic and the pessimistic banks, respectively. When $\bar{\alpha}_m$ is smaller than 9.68%, the owner constitutes the newsvendor as a pure equity firm. Therefore, the equilibrium order quantity is the same regardless of whether the bank is optimistic or pessimistic, i.e., information asymmetry has no effect on operational investment.

Table 2: Equilibrium order quantity and debt to assets ratio as a function of the expected market return. Demand is exponential with mean 10. Optimistic, and pessimistic banks believe that demand is exponential with mean 12 and 8, respectively. $p = 1$, $c = 0.6$, $b = 0.2$, $s = 0.1$, $\alpha = 0.15$, $\tau = 0$. When the bank is pessimistic, the owner chooses pure equity financing if $0 \leq \bar{\alpha}_m \leq 11.85\%$, debt-equity mix if $11.85\% < \bar{\alpha}_m \leq 45.2\%$, and pure debt financing if $\bar{\alpha}_m > 45.2\%$. When the bank is optimistic, the owner chooses pure equity financing if $0 \leq \bar{\alpha}_m \leq 9.68\%$, debt-equity mix if $9.68\% < \bar{\alpha}_m \leq 36.5\%$, and pure debt financing if $\bar{\alpha}_m > 36.5\%$.

	Pessimistic Bank		Optimistic Bank	
$\bar{\alpha}_m$	Order Quantity	Debt/Asset Ratio	Order Quantity	Debt/Asset Ratio
5%	5.30	0	5.30	0
10%	4.74	0	4.51	0.61
25%	3.06	0.53	3.06	0.83
50%	1.38	1	2.07	1

As $\bar{\alpha}_m$ increases, it exceeds the value of $\tilde{\alpha}$ for the optimistic bank. Therefore, when faced with an optimistic bank, the owner finds it attractive to contribute less equity and borrow with bankruptcy risk. Whereas when faced with a pessimistic bank, the owner continues to contribute more equity and have zero borrowing. As a result, for $\bar{\alpha}_m \in [0.0968, 0.1185)$, we obtain a counterintuitive finding

that the equilibrium order quantity is higher when the owner faces a pessimistic bank.

As $\bar{\alpha}_m$ exceeds 11.85%, the newsvendor borrows up to the credit limit regardless of the bank's view. The owner compensates for the tighter credit limit given by the pessimistic bank by contributing more equity in the newsvendor. As a result, the equilibrium order quantity is unaffected by information asymmetry even though financial leverage varies. Finally, when $\bar{\alpha}_m$ is very high (e.g., 50%), we have a pure debt firm in either case, and tighter credit due to information asymmetry depresses operational investment.

4.2 Effectiveness of Asset Based Lending

In practice, asset based lenders use simple rules of thumb, such as historical salvage value, to determine γ. Our model shows that this practice can be improved by tailoring the collateral value of inventory to a firm using objective optimization criteria. Consider the example presented in Table 2. The salvage value s is 0.1 and the purchase cost c is 0.6 for both the optimistic and the pessimistic banks. Thus, the rule of thumb gives $\gamma = \frac{s}{(1+\alpha)c} = 0.15$ (i.e., setting the credit limit $\psi = \frac{s}{1+\alpha}q$). However, the optimal γ varies due to the differences in the banks' beliefs. For instance, when $\bar{\alpha}_m = 0.10$, we find that at equilibrium the pessimistic bank sets $\gamma^*(q)$ equal to 0.39 and $(q, w, x) = (4.74, 0, 2.84)$, whereas the optimistic bank sets it equal to 0.54 and $(q, w, x) = (4.51, 1.46, 1.25)$.

Further insights into the effectiveness of ABL can be obtained from Proposition 1. First, observe that the optimal credit limit, $\psi^*(q)$, has two components, $\frac{s}{1+\alpha}q$ and $\left(c - \frac{s}{1+\alpha}\right)\theta$. The first component is riskless because it is the maximum amount that the firm can borrow and repay with interest in full with probability 1, i.e., even when the realized demand is zero. This result follows from the definition of the bankruptcy threshold, $d_B = \left(\frac{1+\alpha}{p-s}w - \frac{s}{p-s}q\right)^+$. The second component $\left(c - \frac{s}{1+\alpha}\right)\theta$ is risky because, if the firm borrows more than the first component, its loan repayment depends on the realized demand.

Second, only the risky component of the credit limit depends on information asymmetry and the bankruptcy cost. This outcome is intuitive because information asymmetry comes into the play only when the bank evaluates the newsvendor's demand prospects. Lenders' conservative estimates about potential investments during economic downturns correspond to a lower θ value in our model.

This is consistent with the shrinkage of the aggregate loan supply during economic downturns. (See Leary (2009) and references therein for the relationship between economic downturns and loan supply.)

Third, the bank has full control over its maximum loan write-off because it is independent of the newsvendor's inventory q or the true demand distribution F. To see this, note that the bank's maximum loan write-off occurs when the newsvendor borrows up to the credit limit and realizes a demand equal to zero. Thus, it is equal to $(p-s+b)d_B(q,x) = (p-s+b)\bar{G}^{-1}\left[1-\left(\frac{\alpha}{1+\alpha}\right)\frac{p-s}{p-s+b}\right]$. This quantity does not change with q or F. Therefore, even as the amount of the loan may vary with the inventory of the firm or its true demand distribution, the asset based credit limit creates an upper bound on the bank's loan write-off.

Finally, we find that if the firm uses its entire credit limit then the distribution of the bank's loan write-off is also independent of q. This result is shown in the following lemma.

Lemma 3 *If the firm borrows with risk, then it borrows* $w = \frac{s}{1+\alpha}q + \left(c - \frac{s}{1+\alpha}\right)\theta$ *and the probability distribution of the bank's loan write-off is independent of* q*. In other words, let* $\mathcal{W} \equiv (p-s+b)[d_B - \xi]^+$ *denote the bank's loan write-off in case the firm defaults on its loan, and let the maximum of* $\mathcal{W}$ *be* $\mathcal{W}_{max} \equiv (p-s+b)\bar{G}^{-1}\left[1-\left(\frac{\alpha}{1+\alpha}\right)\frac{p-s}{p-s+b}\right]$*. Then*

$$Pr(\mathcal{W} \leq \zeta) = \begin{cases} \bar{F}\left(\bar{G}^{-1}\left[1-\left(\frac{\alpha}{1+\alpha}\right)\frac{p-s}{p-s+b}\right] - \frac{\zeta}{p-s+b}\right) & \text{if } \zeta \leq \mathcal{W}_{max}, \\ 1 & \text{if } \zeta > \mathcal{W}_{max}, \end{cases} \tag{21}$$

which is independent of x*.*

Intuitively, this lemma means that if the firm seeks to stock more inventory, the bank demands the owner to increase her equity investment proportionately so that the credit limit increases but the write-off distribution remains unchanged. As a consequence, contrary to unsecured loans, the riskiness of an asset based loan does not increase in the borrowing amount. Thus, the bank does not need to increase the interest rate to compensate for risk as the amount of borrowing increases.

4.3 Probability of Bankruptcy

The probability of bankruptcy of the firm in our model is as follows:

$$Pr(\text{Bankruptcy}) = \begin{cases} 0 & \text{if } \tilde{\alpha} \text{ exists and } \bar{\alpha}_m \leq \tilde{\alpha}, \\ F\left(\bar{G}^{-1}\left[1 - \left(\frac{\alpha}{1+\alpha}\right)\frac{p-s}{p-s+b}\right]\right) & \text{if } \tilde{\alpha} \text{ does not exit or } \bar{\alpha}_m > \tilde{\alpha}. \end{cases} \tag{22}$$

To see this, note that if the firm borrows with risk, then the credit limit is *always* binding. Thus, bankruptcy occurs if the demand is less than $d_B(q^L(x), x)$. The probability of occurrence of this event is $Pr\left(\xi \leq d_B(\beta x + \theta, x)\right) = F\left(\bar{G}^{-1}\left[1 - \left(\frac{\alpha}{1+\alpha}\right)\frac{p-s}{p-s+b}\right]\right)$. Thus, the probability of bankruptcy depends on the parameters of the newsvendor model, the attractiveness of the external investment alternative, and information asymmetry.

Note that the equilibrium probability of bankruptcy is either zero or a fixed scalar depending on the regime in which the solution lies. The fixed scalar arises because the bank's profit function in (8) is a newsvendor-like formula. That is, similar to the newsvendor model, the optimal solution for the bank fixes the tail probability of demand. In Appendix A, we show that a similar result in which the probability of bankruptcy is either zero or a positive scalar also arises when the bank optimizes the interest rate without imposing a credit limit. Hence, this result is due to the bank's ability to influence the owner's equity investment decision, not due to the bank's lending model.

The bankruptcy formula also suggests hypotheses regarding the drivers of bankruptcy risk at equilibrium. The effect of information asymmetry is captured by the term $F(\bar{G}^{-1}(\cdot))$. It implies that there is a lower probability of bankruptcy if the bank is pessimistic about the demand distribution. This occurs because the credit limit is smaller. Moreover, the value of $\tilde{\alpha}$ is larger so that there is a smaller range of values of $\bar{\alpha}_m$ for which bankruptcy can occur. Conversely, the bankruptcy probability is higher when the bank is optimistic. Besides information asymmetry, we observe that the probability of bankruptcy increases in p, s, and α because the bank is willing to allow the newsvendor to have a higher order quantity as p, s or α increases. It decreases in b because the bank is more conservative when the bankruptcy cost is high.

Additionally, the market return $\bar{\alpha}_m$ and the threshold $\tilde{\alpha}$ affect the bankruptcy probability by determining whether the owner will inject enough equity to avoid borrowing with risk. Holding all other parameters constant, a higher expected return in the market can increase the probabil-

ity of bankruptcy from 0 to $F\left(\bar{G}^{-1}\left[1-\left(\frac{\alpha}{1+\alpha}\right)\frac{p-s}{p-s+b}\right]\right)$ because the owner has other attractive investment alternatives.

4.4 Capital Structure at Equilibrium

Through the interaction of ABL and the newsvendor model, our analysis leads to empirically testable predictions regarding the relationship between a firm's operational characteristics and its capital structure. We present three such predictions. Future empirical research can examine the extent to which capital structure decisions can be explained by operational dynamics.

1. Leverage is negatively correlated with demand uncertainty. The newsvendor model enables us to model the relationship between demand uncertainty and leverage. Demand uncertainty can be measured by the coefficient of variation of demand. When the profit margin is sufficiently high, the equilibrium order quantity increases in demand uncertainty due to a rise in the safety stock. On the other hand, the minimum credit line offered by the bank declines because the bank wants to limit its losses, which are more likely when the demand is more uncertain. An increase in the equilibrium order quantity and a decrease in the minimum credit line imply an increase in the equity investment. Thus, the debt-equity ratio declines. Figure 4(a) illustrates this reasoning.

To the best of our knowledge, no papers use demand uncertainty as a proxy for operational risk within a capital structure framework, but empirical evidence generated by other risk proxies suggests that leverage generally has a negative correlation with risk (Myers 2001). Despite this evidence, the relationship between risk and leverage is ambiguous in the tradeoff theory and the pecking order theory models (Frank and Goyal (2008) and references therein). Hence, our model presents a theoretical link between operational risk and leverage that may help address this gap.

2. There is a non-monotone relationship between profitability and leverage. The equilibrium order quantity increases in the after tax profit margin because the newsvendor provides a more attractive investment opportunity. Meanwhile, both the owner's tendency to inject equity and the bank's tendency to provide a loan increase in the newsvendor's profitability. As shown in Figure 4(b), these dynamics create a U-shaped relationship between profitability and the debt-equity ratio. This result arises because debt and equity increase at different rates, which makes their ratio non-

monotone.

The majority of the tradeoff theory models predict a positive correlation between profitability and leverage, whereas the pecking order theory models predict a negative correlation (Frank and Goyal 2008). Both predictions are at odds with the empirical evidence. For example, the empirical findings of Birge and Xu (2011) verify the existence of a U-shaped relation between profitability and leverage when the profit margin is sufficiently high. Similar to our paper, Birge and Xu provide an operational justification for this observation. They also discuss the shortcomings of the related empirical studies that seek to analyze a seemingly non-monotone relationship using linear regression.

3. The newsvendor model can explain intra-industry differences in leverage. The empirical corporate finance literature shows that capital structure varies both across industries and across firms within the same industry (Leary and Graham 2010). In fact, a substantial amount of unexplained variation in capital structure is firm specific and time invariant (Lemmon et al. 2008). However, the tradeoff theory and the pecking order theory models cannot accurately explain firm specific differences in capital structure (Myers 2001, Frank and Goyal 2008). Our model provides a theoretical framework to justify and analyze these differences. For example, the newsvendor model predicts a positive correlation between inventory liquidation value and leverage. $\frac{s}{(1+\alpha)c}$ in the newsvendor model captures the relative liquidation value of the firm's inventory. The asset based credit limit increases in $\frac{s}{(1+\alpha)c}$ by (10). That is, tangible assets provide more debt capacity, which leads to a high debt-equity ratio for a borrower firm. This prediction is consistent with the empirical evidence suggesting that leverage has a positive correlation with the firm's assets' liquidation value (Harris and Raviv (1991) and references therein).

To sum up, our model's ability to capture the details of operational and financial dynamics in a realistic framework sheds light on some empirical observations that cannot be accurately explained by the mainstream theoretical models. This implication is well aligned with the recent studies showing the superiority of equilibrium models in explaining capital structure choice (e.g., MacKay and Phillips (2005) and references therein).

5 Conclusions

We characterize the equilibrium order quantity, probability of bankruptcy, and capital structure of a firm in a game played between a small business owner and an asset based lender. Our results are driven by the economic considerations of the risk of bankruptcy, the expected return to the owner of the firm, and the credit limit imposed by the bank. Taxation and cost of bankruptcy play a secondary role. In fact, all of our results go through under zero taxes and zero cost of bankruptcy. Moreover, the form of the probability of bankruptcy is the same even without information asymmetry or under interest rate optimization (as shown in Appendix A).

Our paper falls in a rich area of research. Issues examined in this paper can be studied under alternative models, and some aspects of our model can be generalized in future research. For example, players, esp. the owner, may be modeled as expected utility maximizers, and agency issues can be added. It may also be productive to allow trading between the bank and the owner or to replace them with two investor classes. There may be competition among banks, which can affect the equilibrium outcome under information asymmetry. The bank's strategic interactions with the owner can also be modeled. Appendix A provides a starting point for such an analysis by the deriving the optimal lending interest rate, which is a function of $\bar{\alpha}_m$. Another extension is to introduce a joint distribution for ξ and $\bar{\alpha}_m$ to capture the correlation between consumption and macroeconomic conditions. Signaling and firm-bank coordination through a menu of contracts can also be considered. For example, the bank may consider using a signaling mechanism to infer the true demand distribution of a potential borrower. However, designing such a mechanism is not straightforward because firms with bad demand prospects may replicate good firms' actions. As a result, a pooling equilibrium may exist in which order quantities and borrowing amounts become uninformative; see Greenwald and Stiglitz (1990; page 37) for a similar argument. Another potential direction is to extend our model to a multi-period setting in order to capture time-varying bankruptcy risk. Finally, future empirical research may examine the predictions emerging from our paper regarding the links between operational parameters, capital structure and the probability of bankruptcy.

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Figures

Figure 1: Figures (a) and (b) illustrate the optimal collateral of inventory, $\gamma^*(q)$, and the corresponding asset based credit limit, $\psi^*(q) = \gamma^*(q)cq$, as a function of the firm's starting inventory, q, respectively. $p = 2$, $c = 1$, $s = 0.2$, $b = 0.1$, and $\alpha = 0.15$. The bank believes that demand is exponential with mean 100. That is, $\bar{G}(\xi) = \exp(-\xi/100)$ for $\xi \geq 0$.

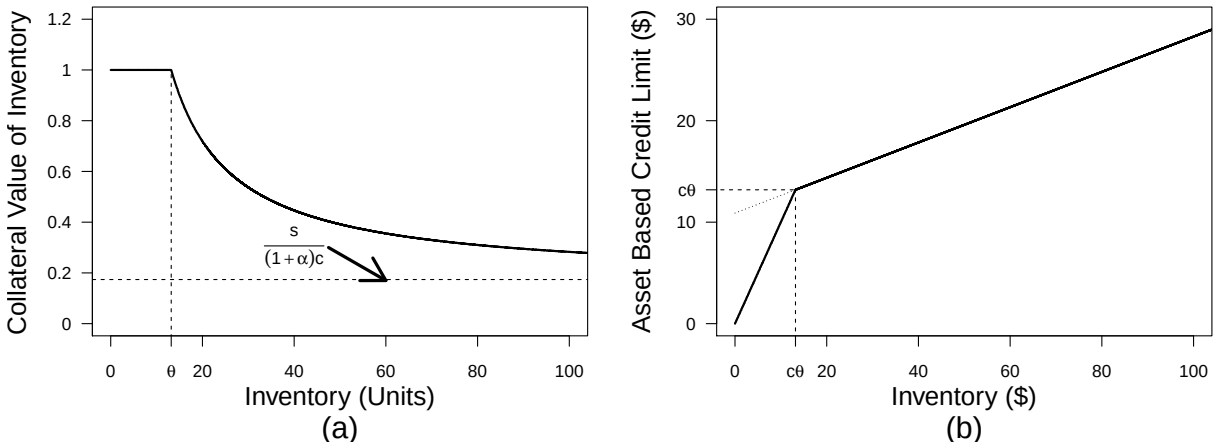


Figure 2: Three cases that can arise as a function of x, q, and ξ. (NB): The newsvendor does not borrow if $q \leq x/c$. It has operating income if $\xi > \frac{c-s}{p-s}q$. Otherwise, it has operating loss. (BWO): The newsvendor borrows without risk if $x/c < q \leq \beta x$. It has operating income if $\xi > d_T = \frac{[(1+\alpha)c-s]q-\alpha x}{p-s}$. Otherwise, it has operating loss. (BWR): The newsvendor borrows with risk if $\beta x < q \leq \beta x + \theta$. It has operating income if $\xi > d_T$. It has operating loss if $\xi < d_T$. It declares bankruptcy if $\xi < d_B = \frac{[(1+\alpha)c-s]q-(1+\alpha)x}{p-s}$. It cannot order more than $\beta x + \theta$ due to the asset based credit limit imposed by the bank. The values of β and θ are defined in Proposition 1.

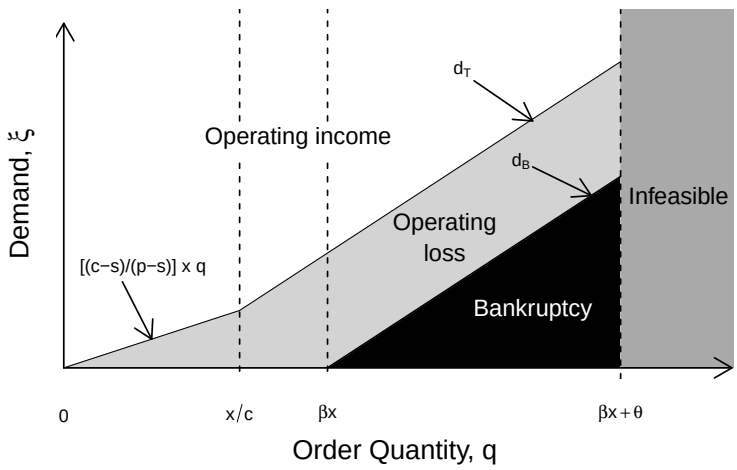
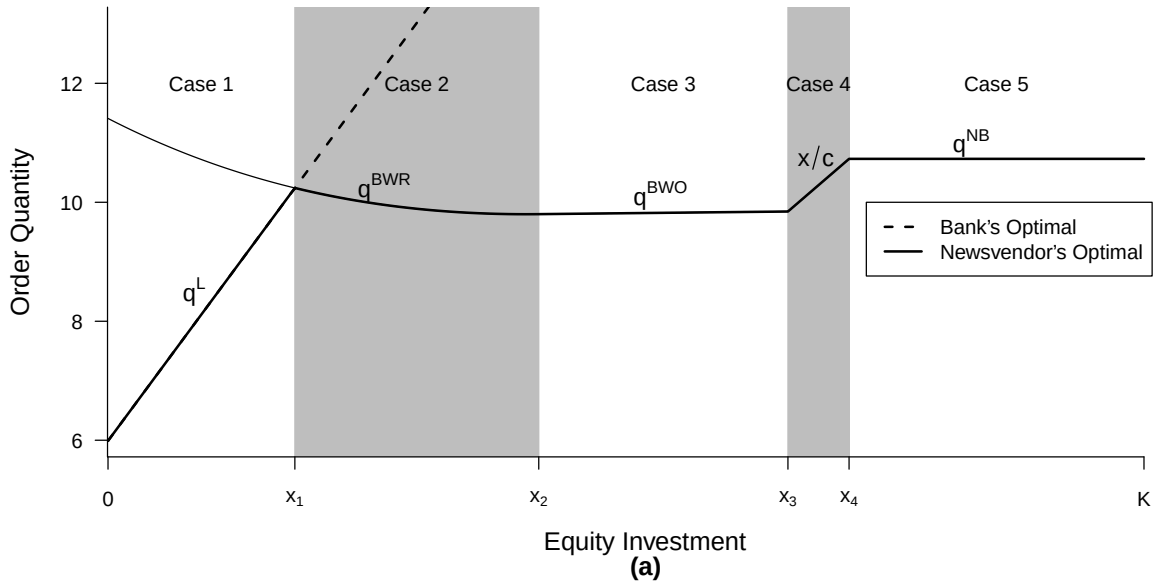
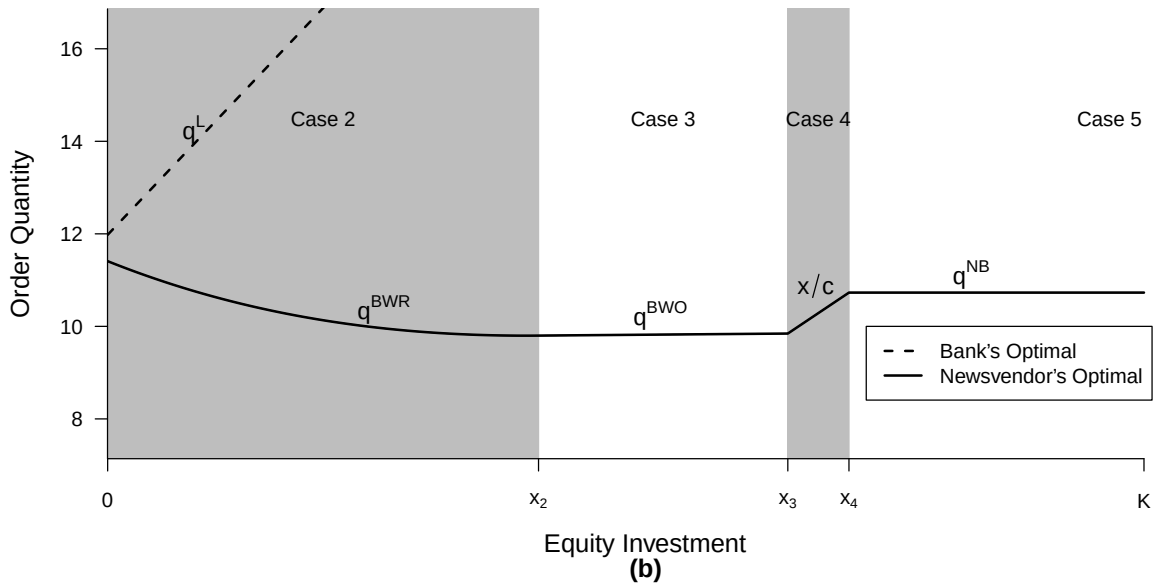


Figure 3: The newsvendor's and the bank's optimal order quantities for different investment levels. In Figure (a), demand is Weibull with $\bar{F}(\xi) = \exp\left(-(\xi/\lambda)^k\right)$. $E_F[\xi] = 10$ and shape parameter $k = 2$. $p = 2$, $c = 1$, $s = 0.4$, $b = 0.2$, $\tau = 0.4$, and $\alpha = 0.10$. The bank believes that demand is Weibull with $E_G[\xi] = 8$ and shape parameter $k = 2$. As a result, $q^{BWR}(0) = 11.41 > q^L(0) = 5.99$, and the credit limit binds for $x \in [0, x_1]$. $x_1 = 2.71$, $x_2 = 6.24$, $x_3 = 9.85$, $x_4 = 10.73$, and $q^{NB} = 10.73$. In Figure (b), all the model parameters are the same as Figure (a) except $E_G[\xi] = 16$. As a result, $q^L(0) = 11.98 > q^{BWR}(0) = 11.41$, and the credit limit never binds.

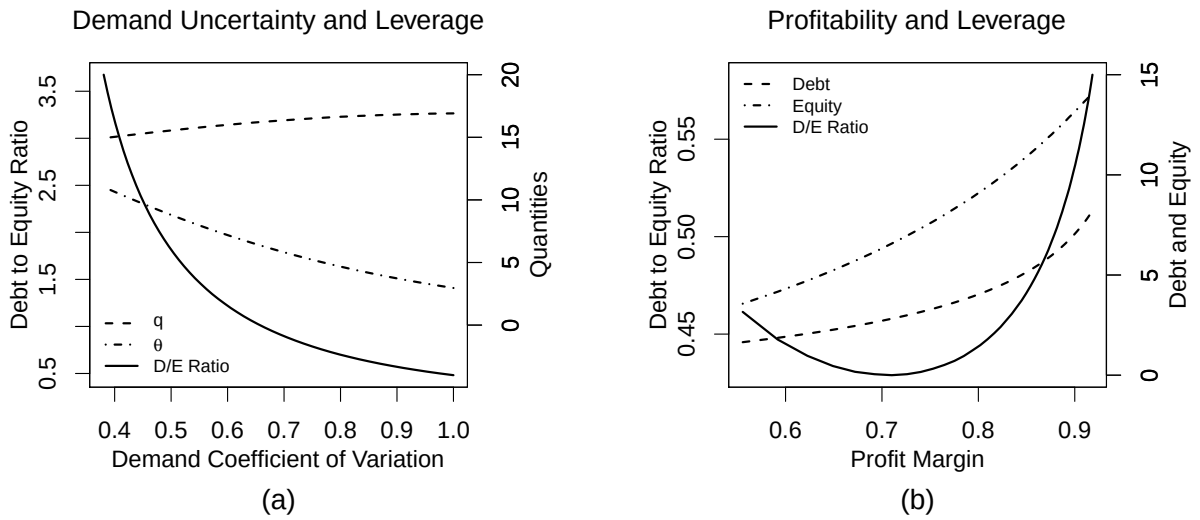


(a)



(b)

Figure 4: The equilibrium capital structure as a function of demand uncertainty and profitability. $\bar{\alpha}_m = 0.15$, $\alpha = 0.10$, $p = 6$, $c = 1$, $s = 0.2$, $b = 0.2$, and $\tau = 0.3$. In Figure (a), demand is Weibull with $E_F[\xi] = 10$, and the bank believes that it is Weibull with $E_G[\xi] = 5$. We vary the shape and scale parameters of F and G to measure the impact of demand variability on the debt-equity ratio. In Figure (b), demand is exponential with $E_F[\xi] = 10$, and the bank believes that it is exponential with $E_G[\xi] = 5$. We vary p between 4 and 8 to measure the impact of firm profitability on the debt-equity ratio. We define the profit margin as $(p - c)/(p - s)$.



Online Appendix - Operational Investment and Capital Structure Under Asset Based Lending

In part A of this appendix, we examine our assumption of a fixed lending interest rate with respect to adverse selection, present alternative models with interest rate optimization, and illustrate that using ABL leads to a higher expected profit for the bank than optimizing the interest rate without a credit limit. In part B, we present the proofs of all propositions and lemmas in the paper.

A. Interest Rate Optimization

Our assumption of fixed interest rate prompts two questions: (i) why does the bank not optimize the interest rate for an individual borrower firm, and (ii) is our result on probability of bankruptcy dependent on the assumption of fixed interest rate. In Section 3, we showed that interest rates are capped for some loans in the economy, and for such loans, the assumption of fixed interest rates is appropriate. In addition, our paper differs from the literature in operations wherein banks optimize interest rates (Dada and Hu 2008) or wherein the interest rate is set to achieve the risk-free rate of return on average (Xu and Birge 2004) for two reasons: equity is a decision variable and there is information asymmetry. These two differences impact our modeling choice of a fixed interest rate as we explain below.

The literature in economics shows that at any given interest rate, under information asymmetry, firms with poor projects (i.e., poor demand prospects in our model) will have a tendency to overborrow compared to firms with good projects. This creates the well known adverse selection problem for the bank because increasing the interest rate can increase the riskiness of the bank's loan portfolio by changing the set of borrowing firms and inducing firms to make riskier investments (Stiglitz and Weiss 1981). (A riskier investment corresponds to a higher order quantity in our model.) As a result, credit rationing, under which some borrowers receive smaller loans than they demand, may exist because it may not be optimal for the bank to increase the interest rate until loan supply equals loan demand. Greenwald and Stiglitz (1990, page 15) explain this market failure (i.e., supply-demand mismatch) by stating that "in credit markets, it is by now well established that lenders who are less well-informed than borrowers about the risk characteristics of the borrower's investment projects may well respond by fixing interest rates and (under certain conditions) rationing credit."

The implication of adverse selection for our model is that the lending interest rate should be optimized for a *population of borrowers*, not for an individual firm. Such a population can be defined based on model primitives, including G, p, c, and s. Therefore, if the bank has the flexibility to optimize the interest rate, then the optimal interest rate will be a function of model primitives, but will not depend on the owner's actions. Hence, α continues to be exogenous for a given firm. This

result arises because making α a function of x, w and/or q amplifies adverse selection by urging firms with good (poor) demand prospects to borrow less (more).

This section explores these implications in detail.

1. We show that increasing the interest rate does not help the bank because it leads to an equilibrium outcome in which the interest rate is very high and only firms with poor demand prospects borrow.

2. We present the bank's interest rate optimization problem in the context of information asymmetry, and validate that the optimal interest rate that maximizes the bank's expected profit under G is a function of model primitives, including $\bar{\alpha}_m$, G, p, and c, but is independent of the owner's and the firm's decisions (i.e., x, w, and q). Thus, the fixed interest rate α in our model can be interpreted as the optimal interest rate for a population of borrowers or an exogenous interest rate imposed by a government agency. This result holds regardless of whether the bank imposes a credit limit or not.

3. We show numerically that imposing a credit limit increases the bank's true expected profit because it prevents over-borrowing. Moreover, if the bank imposes a credit limit then charging an exogenous interest rate that is lower than the optimal interest rate under G increases the bank's true expected profit. However, if the bank does not impose a credit limit then charging an exogenous interest rate that is lower than the optimal interest rate under G decreases the bank's true expected profit.

We perform interest rate optimization in two steps. First, we solve the owner's problem to find the order quantity, equity and debt that maximize her expected profit for a given interest rate α. Finding $(q(\alpha), x(\alpha), w(\alpha))$ allows us to compute the bank's expected profit as a function of α. Then we find the interest rate that maximizes the bank's expected profit. This optimization procedure is similar to the one used by Dada and Hu (2008).

We illustrate interest rate optimization using a population of two borrowers. Suppose there are two owners with two proprietary newsvendor type investment opportunities. Firm i faces random demand ξ_i characterized by F_i, $i \in \{1, 2\}$. The bank cannot distinguish among the firms, and believes that both firms face identically distributed demand functions characterized by G. We refer to G as the bank's belief. $\bar{\alpha}_m$, p, c, and K are identical for both owners and common knowledge to the bank. For analytical tractability, we focus on a special case in which the tax rate τ, bankruptcy cost b and salvage value s are zero. Our results continue to apply when these quantities are positive.

Interest Rate Optimization With A Credit Limit

In this section, we show how the bank can find the optimal α value that maximizes its expected profits under its belief G given the credit limit computed in Proposition 1. Setting $s = b = 0$ and replacing F with G in Table 1 gives us the values of $(q(\alpha), x(\alpha), w(\alpha))$ from the bank's perspective.

We see that a pure equity (PE) firm facing G orders $q^{PE} = \bar{G}^{-1}\left(\frac{(1+\bar{\alpha}_m)c}{p}\right)$. The corresponding expected profit for the owner under the bank's belief G is

$$\Pi_G^{PE} \equiv (1+\bar{\alpha}_m)\left(K - cq^{PE}\right) + p\int_0^{q^{PE}} \bar{G}(\xi)d\xi. \tag{1}$$

Similarly, a firm that uses a mix of debt and equity (DE) orders $q^{DE} = \bar{G}^{-1}\left(\frac{(1+\bar{\alpha}_m)c}{p}\right)$, which is equal to q^{PE}. The corresponding expected profit for the owner under G is equal to

$$\Pi_G^{DE}(\alpha) \equiv (1+\bar{\alpha}_m)\left(K - c(q^{PD} - \theta)\right) + p\int_{\frac{(1+\alpha)c}{p}\theta}^{q^{PD}} \bar{G}(\xi)d\xi, \tag{2}$$

where $\theta = \frac{p}{(1+\alpha)c}\bar{G}^{-1}\left(\frac{1}{1+\alpha}\right)$. Lastly, a pure debt (PD) firm orders θ, and the owner's expected profit under G is

$$\Pi_G^{PD}(\alpha) \equiv (1+\bar{\alpha}_m)K + p\int_{\frac{(1+\alpha)c}{p}\theta}^{\theta} \bar{G}(\xi)d\xi. \tag{3}$$

Under the bank's belief G, the firm should borrow if α is such that $\max\{\Pi_G^{PD}(\alpha), \Pi_G^{DE}(\alpha)\} \geq \Pi_G^{PE}$. This inequality is a participation constraint, which ensures that the owner is better off by using pure debt financing or a mix of debt and equity rather than pure equity financing. Following Proposition 1 and Table 1, if the firm borrows, the bank's expected profit under G will be equal to $p\int_0^{\frac{(1+\alpha)c}{p}\theta} \bar{G}(\xi)d\xi - c\theta$. Hence, the bank solves the following interest rate optimization problem:

$$\max_{\alpha\geq 0} \quad p\int_0^{\frac{(1+\alpha)c}{p}\theta} \bar{G}(\xi)d\xi - c\theta \quad \text{s.t.} \quad \max\{\Pi_G^{PD}(\alpha), \Pi_G^{DE}(\alpha)\} \geq \Pi_G^{PE}.$$

We observe that the optimal interest rate under a credit limit α_C^* is a function of model primitives, including G, because the true distribution $\bar{F}_i$, actual order quantity or equity investment do not appear in the above optimization problem. Hence, the optimal interest rate $\alpha_C^*(p, c, \bar{\alpha}_m, G)$ is exogenously determined for a potential borrower.

When salvage value, taxation, and bankruptcy costs are added to the model, the equilibrium order quantity and starting equity become implicit functions of the interest rate. The bank can find the firm's best response, which consists of an order quantity $q(\alpha; G)$ and a starting equity $x(\alpha; G)$. In fact, replacing $\bar{F}$ terms with $\bar{G}$ in Propositions 4 and 5 gives the potential equilibrium outcomes under the bank's belief for a given α. Once the bank computes $x(\alpha; G)$ and $q(\alpha; G)$ for a continuum of α values, it can find the optimal α value that maximizes its expected profits. Despite being notationally cumbersome, this optimization exercise still has a single decision variable α. Hence, α_C^* is still exogenously determined, but becomes a function of additional model primitives, including τ, b, and s. The optimal solution under taxation, bankruptcy costs, and a positive salvage value, $\alpha_C^*(p, c, s, \tau, b, \bar{\alpha}_m, G)$, corresponds to the optimal interest rate for the bank's problem we present in Section 3.1.

Interest Rate Optimization Without A Credit Limit

Another lending model for the bank is unsecured lending without a credit limit. The interest rate is the only decision variable for the bank.

The newsvendor-bank interaction in this model is similar to the case that we illustrate in Figure 3(b) in the paper. That is, for a given starting equity x and interest rate α, the firm's order quantity is not restricted by a credit limit. The absence of the credit limit implies that if $x \in [0, x_2]$ then the firm borrows with risk, and the order quantity solves (14) in the paper. Furthermore, as we show in Lemma 2, the owner's profit function is convex in the interval in which the firm borrows with risk without a binding credit limit. This convexity result implies that in the absence of a credit limit, if the owner decides to borrow then it is optimal to create a pure debt firm.

Continuing the special case with $\tau = 0$ and $s = b = 0$, the bank believes that if an owner chooses to borrow, she creates a pure debt firm, and the order quantity solves $\tilde{q}^{PD} = \bar{G}^{-1}\left(\frac{(1+\alpha)c}{p}\bar{G}\left(\frac{(1+\alpha)c}{p}\tilde{q}^{PD}\right)\right)$. This result follows from setting τ, b, s equal to zero in (14) in the paper. As a result, the owner's expected ending cash position under G is

$$\tilde{\Pi}_G^{PD}(\alpha) \equiv (1+\bar{\alpha}_m)K + p\int_{\frac{(1+\alpha)c}{p}\tilde{q}^{PD}}^{\tilde{q}^{PD}} \bar{G}(\xi)d\xi. \tag{4}$$

Another alternative for the owner is to create a pure equity firm. This case is identical to the pure equity case presented under a credit limit. That is, a firm facing G orders $\tilde{q}^{PE} = \bar{G}^{-1}\left(\frac{(1+\bar{\alpha}_m)c}{p}\right)$, and the owner's expected ending cash position is

$$\tilde{\Pi}_G^{PE} \equiv (1+\bar{\alpha}_m)\left(K - \tilde{q}^{PE}\right) + p\int_0^{\tilde{q}^{PE}} \bar{G}(\xi)d\xi. \tag{5}$$

The bank believes that the owner creates a pure debt firm if $\tilde{\Pi}_G^{PD}(\alpha) \geq \tilde{\Pi}_G^{PE}$. Otherwise, she creates a pure equity firm. If the owner creates a pure debt firm, then the bank's expected profit under G will be equal to $p\int_0^{\frac{(1+\alpha)c}{p}\tilde{q}^{PD}} \bar{G}(\xi)d\xi - c\tilde{q}^{PD}$. Hence, the bank solves the following interest rate optimization problem:

$$\begin{aligned} \max_{\alpha \geq 0} \quad & p\int_0^{\frac{(1+\alpha)c}{p}\tilde{q}^{PD}} \bar{G}(\xi)d\xi - c\tilde{q}^{PD} \\ \text{s.t.} \quad & \tilde{\Pi}_G^{PD}(\alpha) \geq \tilde{\Pi}_G^{PE}. \end{aligned}$$

Similar to the interest rate optimization with a credit limit, this problem has a single decision variable α, and the optimal interest rate is a function of model primitives, including G. Hence, the optimal interest rate under unconstrained borrowing α_U^* is exogenously determined for a potential borrower.

The equilibrium probability of bankruptcy of a firm i is either zero or a positive scalar because firm i is either a pure equity or a pure debt firm. If it is a pure debt firm, then the equilibrium order quantity $\tilde{q}_i^{PD}$ solves $\tilde{q}_i^{PD} = \bar{F}_i^{-1}\left(\frac{(1+\alpha_U^*)c}{p}\bar{F}_i\left(\frac{(1+\alpha_U^*)c}{p}\tilde{q}_i^{PD}\right)\right)$. Hence, the probability of

bankruptcy is equal to $F\left(\frac{(1+\alpha_U^*)c}{p}\tilde{q}_i^{PD}\right)$. Obtaining a fixed probability of bankruptcy result under interest rate optimization without a credit limit shows the robustness of the equilibrium probability of bankruptcy result we present in Section 4.3.

Once again, adding taxation and bankruptcy costs to the model makes the optimal interest a function of additional model primitives, but α_U^* remains exogenous.

It is worthwhile to note that this optimization procedure is similar to the one presented by Dada and Hu (2008). Using the two-step procedure described above, Dada and Hu find the equilibrium interest rate and order quantity in a game played between a newsvendor firm and a profit maximizing bank. Starting equity is given, and the bank, which has full information about the firm's demand prospects, determines the interest rate that maximizes its expected profit. In their model, adverse selection does not exist because there is no information asymmetry. Furthermore, the equilibrium interest rate is a function of the firm's starting equity because starting equity is not a decision variable. As shown above, allowing the owner to change her firm's equity leads to a pure debt or a pure equity firm under interest rate optimization without a credit limit because the owner's objective function is convex in equity. This result holds even under full information (i.e., even when $F = G$), which shows the necessity of imposing a credit limit to obtain a realistic capital structure outcome.

Instead of the two-step interest optimization procedure originating from Dada and Hu (2008), we could directly optimize the bank's expected profit function, which appear in (9) in the paper, with respect to α for a given (q, w, x). Unfortunately, this problem is not well defined. To see this, consider the bank's optimization problem in terms of α:

$$\max_{\alpha} \quad \alpha w - (p - s + b)\int_0^{d_B} G(\xi)d\xi.$$

This function is concave in α if $d_B > 0$. The optimal solution is $\alpha = \frac{1}{w}[sq + (p-s)G^{-1}(\frac{p-s}{p-s+b})] - 1$. For simplicity, let $F = G$, i.e., there is no information asymmetry. Now, input the optimal value of α into the owner's optimization problem. We find that $d_B = F^{-1}(\frac{p-s}{p-s+b})$, $d_T = F^{-1}(\frac{p-s}{p-s+b}) + x/(p-s)$, and the the firm's expected profit is monotone increasing in q. Moreover, (i) the interest rate α is decreasing in the amount of debt w, (ii) the bank's expected profit is linearly increasing in α if there is no bankruptcy risk, (iii) the optimal interest rate becomes unbounded if $b = 0$. These impractical results arise because optimizing the bank's expected profit with respect to α for a given (q, w, x) ignores the owner's response to the changes in α. A similar issue also arises in Dada and Hu (2008) under full information. See page 571 for their explanation regarding the necessity of finding the firm's best response in optimizing the bank's expected profits.

Implications of Adverse Selection

With A Credit Limit While optimizing the interest rate for a population of borrowers under a credit limit increases the bank's expected profit under its belief, it can still lead to lower profits due to asymmetric information. We illustrate the existence of adverse selection with a numerical

example. To be consistent with the adverse selection literature, we assume that $\int_0^x \bar{F}_2(\xi)d\xi \leq \int_0^x \bar{G}(\xi)d\xi \leq \int_0^x \bar{F}_1(\xi)d\xi$ for all $x \geq 0$ with $E_G[\xi] = E_{F_1}[\xi] = E_{F_2}[\xi]$. That is, the bank's belief is a mean preserving spread of firm 1's demand, and firm 2's demand is a mean preserving spread of the bank's belief. This ordering implies that both firms have the same expected demand, but firm 2 has more demand uncertainty. Furthermore, the bank overestimates firm 1's risk, and underestimates firm 2's risk. The use of mean preserving spread for risk assessment originates from Rothschild and Stiglitz (1970), and is common in the literature (e.g., Stiglitz and Weiss 1981).

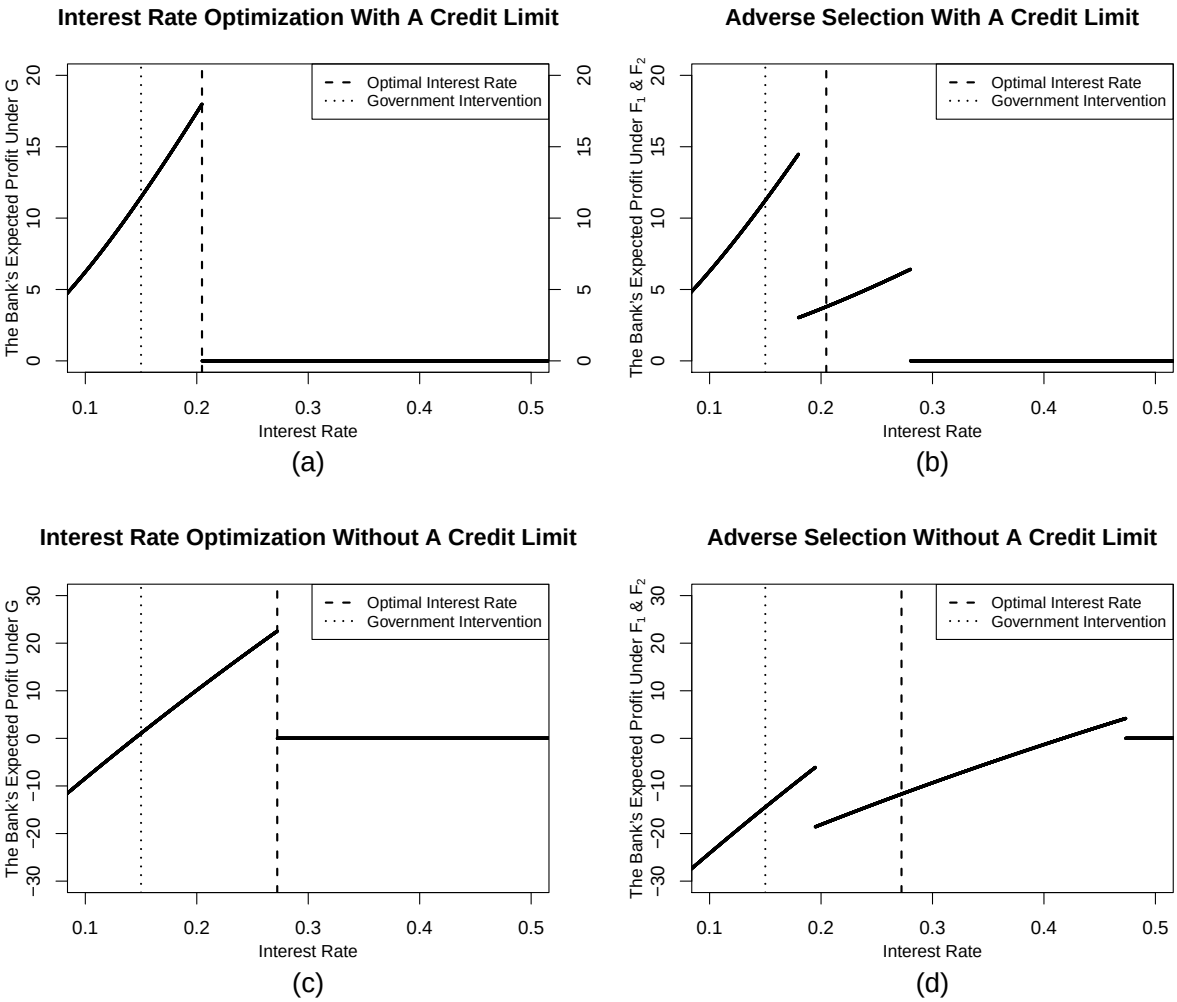
Recall that the bank optimizes the interest rate based on its belief G. For any given α, including $\alpha_C^*(p, c, s, \tau, b, \bar{\alpha}_m, G)$, cost of borrowing is relatively high for firm 1, which has less risky demand prospects, compared to firm 2, which has more risky demand prospects. As a result, owner 1 might choose to create a pure equity firm, whereas firm 2 might be financed with pure debt or debt-equity mix. Figure 1 illustrates this well known adverse selection problem. Figure 1(a) shows the bank's interest rate optimization problem. The participation constraint is binding, and $\alpha_C^* = 21\%$ maximizes the bank's expected profits under G. Figure 1(b) shows the bank's true expected profit as a function of the interest rate. Since α_C^* is too high for firm 1, only firm 2 borrows when $\alpha = \alpha_C^*$, and the bank's expected profit is 3.79. As a result, the bank loses the opportunity to lend to firm 1 because it sets the interest rate too high. This example illustrates the danger of interest rate optimization under information asymmetry. Please see (Stiglitz and Weiss 1981: page 397, Figure 3) for a similar result in the adverse selection literature.

A government intervention can help the bank increase its profit because a government mandate that forces the bank to charge less than α_C^* can mitigate adverse selection by increasing the quality of borrowers. For instance, in the example we present in Figure 1(b), if the government forced the bank to charge no more than 15%, then the bank would charge 15% and both firms would borrow. As a result, the bank's expected profit would be equal to 11.24, which is higher than its expected profits at $\alpha = \alpha_C^*$. Settings similar to this example are used in macroeconomics to justify government control on lending terms, e.g., Ordover and Weiss (1981) and Mankiw (1986). Such government practices are also supported by the data we present in Section 3 of the paper.

Stiglitz and Weiss (1981) also show that credit is rationed at equilibrium. In our example, the credit limit serves as a credit rationing mechanism. It is binding for firm 1 if $\alpha \leq 18\%$ and for firm 2 if $\alpha \leq 28\%$. The next section shows that interest rate optimization in the absence of a credit limit can have disastrous consequences and that imposing a credit limit mitigates those consequences by preventing over-borrowing.

Without A Credit Limit Interest rate optimization in the absence of a credit limit suffers not only from adverse selection but also because the bank does not have a credit allocation device to ration credit. Figures 1(c) and 1(d) show the pitfall of this lending mechanism with a numerical example. We generate Figures 1(c) and 1(d) with the parameter values we used to generate Figures 1(a) and 1(b). Figure 1(c) shows the bank's interest rate optimization problem. The participation constraint is binding, and $\alpha_U^* = 27\%$ maximizes the bank's expected profits under G. Figure 1(d)

Figure 1: The bank's interest rate optimization with and without a credit limit. There are two firms in the economy. Firm 1's demand is Weibull with mean $\mu_1 = 100$ and parameters $k_1 = 2$ and $\lambda_1 = \mu_1/\Gamma(1 + 1/k_1)$. Firm 2's demand is Weibull with mean $\mu_2 = 100$ and parameters $k_2 = 1$ and $\lambda_2 = \mu_2/\Gamma(1 + 1/k_2)$. The bank believes that both firms have Weibull demand with mean $\mu_B = 100$ and parameters $k_B = 1.5$ and $\lambda_B = \mu_B/\Gamma(1 + 1/k_B)$. Hence, $\bar{F}_i(x) = \exp(-(x/\lambda_i)^{k_i})$ for $i \in \{1, 2\}$, and $\bar{G}(x) = \exp(-(x/\lambda_B)^{k_B})$. $p = 2.5$, $c = 1$, $\bar{\alpha}_m = 0.12$, $\tau = 0$, and $b = s = 0$. Figures (a) and (b) show the bank's interest rate optimization and adverse selection under a credit limit, respectively. Figure (a) demonstrates that if the bank imposes a credit limit, $\alpha^*_C = 0.21$ maximizes its expected profits under G. Figure (b) shows adverse selection under a credit limit by showing the bank's expected profits under $F_{i\in\{1,2\}}$ as a function of the interest rate. Firm 1 borrows if $\alpha \leq 18\%$. Firm 2 borrows if $\alpha \leq 28\%$. Figures (c) and (d) show the bank's interest rate optimization and adverse selection without a credit limit, respectively. Figure (c) demonstrates that if the bank does not impose a credit limit, $\alpha^*_U = 0.27$ maximizes the bank's expected profits under G. Figure (d) shows adverse selection in the absence of a credit limit by showing the bank's expected profits under $F_{i\in\{1,2\}}$ as a function of the interest rate. Firm 1 borrows if $\alpha \leq 19.5\%$. Firm 2 borrows if $\alpha \leq 47.3\%$.



shows the bank's true expected profit as a function of the interest rate. Once again, α_U^* is too high firm 1. Hence, only firm 2 borrows when $\alpha = \alpha_U^*$, and the bank's profit is -11.70. The bank loses money in expectation because it cannot prevent firm 2 from over-borrowing. A government regulation that forces the bank to lend at a lower interest rate amplifies over-borrowing. For example, if the bank is forced to charge less than 15%, then it charges 15% with the belief that its expected profit will be equal to 1.03. Both firms borrow at $\alpha = 15\%$. At $\alpha = 15\%$, the true expected profits from firm 1 and firm 2 are 8.38 and -22.82, respectively. Hence, the bank's total expected profit under $F_{i\in\{1,2\}}$ is -14.44. Once again, this result arises because firm 2 borrows excessively, and plays a risk free gamble with the bank's money.

Comparing the true expected profits with and without a credit limit illustrates the benefit of imposing a credit limit. Figure 1(d) shows that the bank loses money in expectation if $\alpha \leq 41.7\%$. Put differently, for any given interest rate that is below 41.7%, the bank's true expected profit with a credit limit is higher than its true expected profit in the absence of a credit limit. For example, at $\alpha = 15\%$, the bank's true expected profits with and without a credit limit are 11.24 and -14.44, respectively. Hence, imposing a credit limit mitigates adverse selection regardless of whether interest rates are optimized or exogenously determined, whereas using the interest rate as a single instrument can lead to excessive losses.

B. Proofs

Proposition 1 *Let* $\theta = \frac{p-s}{(1+\alpha)c-s}\bar{G}^{-1}\left(1-\left(\frac{\alpha}{1+\alpha}\right)\frac{p-s}{p-s+b}\right)$. *The optimal collateral value of inventory,* $\gamma^*(q)$ *equals 1 if* $q \leq \theta$, *and equals* $\frac{s}{(1+\alpha)c} + \left(1 - \frac{s}{(1+\alpha)c}\right)\frac{\theta}{q}$ *if* $q > \theta$. *The corresponding asset based credit limit offered by the bank is equal to* $\psi^*(q) = \gamma^*(q)cq$.

Proof of Proposition 1. Taking a derivative of the bank's objective function, $E_G[\kappa(q, \gamma cq, \xi)]$, with respect to γ and setting equal to zero gives $\gamma^*(q)$ as one can show that the bank's expected profit under the newsvendor bankruptcy is concave in γ for a given q. Multiplying $\gamma^*(q)$ by cq gives $\psi^*(q)$.

Proposition 2 *Let* $q^{BWR}(x), q^{BWO}(x), q^{NB}$ *be order quantities defined by:*

$$q^{BWR}(x) = \bar{F}^{-1}\left(\frac{(1+\alpha)c-s}{(1-\tau)(p-s)}\left[\bar{F}\left(d_B\right) - \tau\bar{F}\left(d_T\right)\right]\right), \tag{6}$$

$$q^{BWO}(x) = \bar{F}^{-1}\left(\frac{(1+\alpha)c-s}{(1-\tau)(p-s)}\left[1 - \tau\bar{F}\left(d_T\right)\right]\right), \tag{7}$$

$$q^{NB} = \bar{F}^{-1}\left(\frac{c-s}{(1-\tau)(p-s)}\left[1 - \tau\bar{F}\left(\frac{c-s}{p-s}q^{NB}\right)\right]\right). \tag{8}$$

The optimal order quantity for the newsvendor without a credit limit, $\tilde{q}(x)$, *is given by* $q^{BWR}(x)$ *if* $0 \leq x < x_2$, $q^{BWO}(x)$ *if* $x_2 \leq x < x_3$, x/c *if* $x_3 \leq x \leq x_4$, q^{NB} *if* $x > x_4$, *where* $x_2 \leq x_3 \leq x_4$ *are cutoff values of the newsvendor's equity. These cutoff values are uniquely defined by* $x_2 = \frac{(1+\alpha)c-s}{1+\alpha}q^{BWR}(x_2)$, $x_3 = cq^{BWO}(x_3)$, $x_4 = cq^{NB}$.

Proof of Proposition 2. The first derivative of $E_F\left[\pi^{NB}\right]$ with respect to q is

$$(1-\tau)(p-s)\bar{F}(q)-(c-s)\left[1-\tau\bar{F}\left(\frac{c-s}{p-s}q\right)\right],$$

whereas the second derivative is $-(1-\tau)(p-s)f(q)-\tau\frac{(c-s)^2}{p-s}f\left(\frac{c-s}{p-s}q\right)\leq 0$. Hence, $E\left[\pi^{NB}\right]$ is concave. Solving the first order condition for this subscenario gives q^{NB}. Then x_4 is equal to cq^{NB}. The first derivative of $E_F[\pi^{BWO}]$ with respect to q is

$$(1-\tau)(p-s)\bar{F}(q)-\left[(1+\alpha)c-s\right]\left(1-\tau\bar{F}\left(d_T(q,x)\right)\right),$$

whereas the second derivative is $-(1-\tau)(p-s)f(q)-\tau\frac{[(1+\alpha)c-s]^2}{p-s}f\left(d_T(q,x)\right)\leq 0$. Hence, $E_F\left[\pi^{BWO}\right]$ is also concave. Solving the first order condition gives $q^{BWO}(x)$. x_3 can be obtained by setting $x=cq^{BWO}(x)$, and solving the first order condition. $(1-\tau)(p-s)>(1+\alpha)c-s$ guarantees the existence of $q^{BWO}(x)$, $q^{BWR}(x)$, and q^{NB}.

We need to show that $x_3\leq x_4$ to define the case in which the newsvendor does not borrow, but uses all of her cash for procurement (i.e., $q=x/c$). Observe that $q^{BWO}(x_3)$ is obtained by solving

$$(1-\tau)(p-s)\bar{F}(q)-\left[(1+\alpha)c-s\right]\left(1-\tau\bar{F}\left(\frac{c-s}{p-s}q\right)\right)=0, \tag{9}$$

whereas $q^{NB}(x_4)$ is obtained by solving the same equation in which α is replaced with 0. Implicit differentiation of q with respect to α in (9) gives

$$\frac{dq}{d\alpha}=-\frac{(p-s)c\left(1-\tau\bar{F}\left(\frac{c-s}{p-s}q\right)\right)}{(1-\tau)(p-s)^2f(q)+\tau[(1+\alpha)c-s](c-s)f\left(\frac{c-s}{p-s}q\right)}<0.$$

Hence, $q^{BWO}(x_3)<q^{NB}(x_4)$ because $\alpha>0$, which implies that $x_3<x_4$.

The first derivative of $E_F[\pi^{BWR}]$ with respect to q is

$$(1-\tau)(p-s)\bar{F}(q)+\left[(1+\alpha)c-s\right]\left(\tau\bar{F}(d_T)-\bar{F}(d_B)\right).$$

Then the second derivative is

$$\frac{d^2E_F[\pi^{BWR}]}{dq^2}=-(1-\tau)(p-s)f(q)+\frac{[(1+\alpha)c-s]^2}{p-s}\left(f(d_B)-\tau f(d_T)\right). \tag{10}$$

$q^{BWR}(x)$ must satisfy the first order condition. Therefore, at $q=q^{BWR}(x)$

$$\begin{aligned}
\frac{d^2E_F[\pi^{BWR}]}{dq^2}&=(1-\tau)(p-s)\bar{F}(q)\left(\frac{-(1-\tau)(p-s)f(q)+\frac{[(1+\alpha)c-s]^2}{p-s}\left(f(d_B)-\tau f(d_T)\right)}{(1-\tau)(p-s)\bar{F}(q)}\right)\\
&=(1-\tau)(p-s)\bar{F}(q)\left(-z(q)+\frac{(1+\alpha)c-s}{p-s}\frac{f(d_B)-\tau f(d_T)}{\bar{F}(d_B)-\tau\bar{F}(d_T)}\right)\\
&<(1-\tau)(p-s)\bar{F}(q)\left(-z(q)+\frac{(1+\alpha)c-s}{p-s}\left(\frac{f(d_B)}{\bar{F}(d_B)-\tau\bar{F}(d_T)}\right)\right)\\
&<(1-\tau)(p-s)\bar{F}(q)\left(-z(q)+\frac{(1+\alpha)c-s}{(1-\tau)(p-s)}z(d_B)\right)\\
&<0
\end{aligned}$$

where z is the hazard rate function. $z(q) > \frac{(1+\alpha)c-s}{(1-\tau)(p-s)} z(d_B)$ because ξ is IFR and $\frac{(1+\alpha)c-s}{(1-\tau)(p-s)} < 1$. x_2 can be obtained by solving the first order condition of the borrowing with risk case after setting $q = \frac{1+\alpha}{(1+\alpha)c-s} x$.

Lastly, we need to show that $x_2 \leq x_3$. (7) implies that x_2 and x_3 solve

$$(1-\tau)(p-s)\bar{F}\left(\frac{1+\alpha}{(1+\alpha)c-s}x_2\right) + \tau\bar{F}\left(\frac{x_2}{p-s}\right) = (1+\alpha)c - s \tag{11}$$

and

$$(1-\tau)(p-s)\bar{F}\left(\frac{x_3}{c}\right) + \tau\bar{F}\left(\frac{c-s}{p-s}\left(\frac{x_3}{c}\right)\right) = (1+\alpha)c - s, \tag{12}$$

respectively. A pairwise comparison of the scalars multiplied by x_2 and x_3 in (11) and (12) indicates that $\frac{1+\alpha}{(1+\alpha)c-s} \geq \frac{1}{c}$ and $\frac{1}{p-s} \geq \frac{c-s}{(p-s)c}$. Hence, $x_2 \leq x_3$ as at least one of the equations would be violated otherwise.

Proposition 3 *$\tilde{q}(x)$ is continuous in x. It decreases in x when x is sufficiently small, increases in x for $x_2 \leq x \leq x_4$ and is constant for $x \geq x_4$.*

Proof of Proposition 3. The continuity of $\tilde{q}(x)$ follows by taking limits from the left and the right at the breakpoints x_2, x_3 and x_4. For example, at x_2, the newsvendor switches from (BWR) to (BWO). Therefore, the bankruptcy threshold d_B approaches zero as x approaches x_2 from below. Mathematically, $\lim_{x\uparrow x_2} d_B\left(q^{BWR}(x), x\right) = 0$, which implies that

$$\lim_{x\uparrow x_2} \frac{(1+\alpha)c-s}{(1-\tau)(p-s)}\bar{F}\left(q^{BWR}(x_2)\right) = \lim_{x\uparrow x_2} \bar{F}\left(d_B\right) - \tau\bar{F}\left(d_T\right) = 1 - \tau\bar{F}\left(d_T\right).$$

Therefore, $\lim_{x\uparrow x_2} q^{BWR}(x_2) = q^{BWO}(x_2)$. Similar analysis shows the continuity of $\tilde{q}(x)$ at the other cutoff points.

To show that $\tilde{q}(x)$ is decreasing in x for small values of x, we apply the Implicit Function Theorem to (6). We get

$$\frac{dq^{BWR}(x)}{dx} = \frac{\left[(1+\alpha)c-s\right]\left[(1+\alpha)f(d_B) - \tau\alpha f(d_T)\right]}{((1+\alpha)c-s)^2(f(d_B) - \tau f(d_T)) - (1-\tau)(p-s)^2 f\left(q^{BWR}\right)}. \tag{13}$$

The denominator in this equation is equal to $(p-s)\frac{d^2E_F[\pi^{BWR}(x)]}{dq^2}\Big|_{q=q^{BWR}(x)}$, which is shown to be negative in the proof of Proposition 2. The numerator is positive at $x = 0$ because $d_T(q,0) = d_B(q,0)$. Thus, (13) is negative when $x = 0$. Since the function is continuously differentiable, it follows that $\tilde{q}(x)$ decreases in x when x is sufficiently small.

$\tilde{q}(x)$ is increasing in x for $x \in [x_2, x_3]$ because $E_F[\pi^{BWO}]$ is supermodular, i.e.,

$$\frac{\partial^2 E_F[\pi^{BWO}]}{\partial q \partial x} = \alpha\tau\frac{(1+\alpha)c-s}{(p-s)^2} f\left(d_T\right) \geq 0.$$

For $x \in [x_3, x_4]$, we have $\tilde{q}(x) = x/c$, which implies that $\tilde{q}(x)$ increases in $x \in [x_3, x_4]$. Lastly, for $x \geq x_4$, $\tilde{q}(x) = q^{NB}$, which does not vary with x.

Lemma 1 *If $q^{BWR}(0) \leq q^L(0)$ then the credit limit is never binding. If $q^{BWR}(0) > q^L(0)$, then there exists an equity value x_1 such that $0 < x_1 < x_2$ and $q^{BWR}(x_1) = q^L(x_1)$ and the credit limit is binding for $x \in [0, x_1]$. The value of x_1 is given by*

$$(1-\tau)(p-s)\bar{F}(\beta x_1+\theta)+\tau[(1+\alpha)c-s]\bar{F}\left(\frac{x_1+[(1+\alpha)c-s]\theta}{p-s}\right) = [(1+\alpha)c-s]\bar{F}\left(\frac{(1+\alpha)c-s}{p-s}\theta\right). \tag{14}$$

Proof of Lemma 1. $q^{BWR}(x)$ and $q^L(x)$ are both continuous functions. First, we show that $q^{BWR}(x_2) < q^L(x_2)$. To see this, note that $x_2 = q^{BWR}(x_2)/\beta$, whereas $q^L(x_2) = \beta x_2 + \theta = q^{BWR}(x_2) + \theta > q^{BWR}(x_2)$. Therefore, it follows that if $q^{BWR}(0) > q^L(0)$, then there exists $0 < x_1 < x_2$ such that $q^{BWR}(x_1) = q^L(x_1)$. Substituting $q^{BWR}(x_1) = q^L(x_1) = \beta x_1 + \theta$ in (6) gives (14). x_1 is unique because the left hand side of (14) increases in x. For the other direction, suppose $q^L(0) \leq q^{BWR}(0)$, and the two functions intersect. The intersection point must be unique because, as we explained above, it has to satisfy (14). If the intersection point is unique, then the derivatives of the two functions at that point must be equal to each other (i.e., the derivative of $q^L(x) - q^{BWR}(x)$ must be zero). From (13) and Proposition 1, $\frac{dq^L(x)}{dx} = \frac{dq^{BWR}(x)}{dx}$ is equivalent to

$$\beta = \frac{[(1+\alpha)c-s]\,[(1+\alpha)f(d_B) - \tau\alpha f(d_T)]}{((1+\alpha)c-s)^2(f(d_B)-\tau f(d_T)) - (1-\tau)(p-s)^2 f\left(q^{BWR}\right)},$$

which can be written as

$$\tau((1+\alpha)c-s)^2 f(d_T) + (1-\tau)(1+\alpha)(p-s)^2 f(q^{BWR}) = 0,$$

which cannot hold because ξ has an IFR distribution. Therefore, the two functions cannot intersect if $q^L(0) \leq q^{BWR}(0)$.

Lemma 2 *$\Pi^S(x)$ is convex in x for $x \in (x_1, x_2)$. Thus, borrowing with risk but ordering less than the bank's optimal order quantity q^L cannot arise in equilibrium.*

Proof of Lemma 2. The order quantity in Case 2, $q^{BWR}(x)$, solves

$$(1-\tau)(p-s)\bar{F}(q^{BWR}(x)) + [(1+\alpha)c-s]\left(\tau\bar{F}(d_T) - \bar{F}(d_B)\right) = 0, \tag{15}$$

and the owner solves

$$\Pi_2^{S^*} = \max_{x\in[x_1,x_2]} \Pi_2^S(x) = \max_{x\in[x_1,x_2]} (1+\bar{\alpha}_m)(K-x) + (p-s)\left(\int_{d_B}^{q^{BWR}(x)} \bar{F}(\xi)d\xi - \tau\int_{d_T}^{q^{BWR}(x)} \bar{F}(\xi)d\xi\right)$$

The first derivative of $\Pi_2^S(x)$ with respect to x is

$$\begin{aligned}\frac{d\Pi_2^S(x)}{dx} &= -(1+\bar{\alpha}_m) + (1+\alpha)\bar{F}(d_B) - \tau\alpha\bar{F}(d_T)\\ &= -(1+\bar{\alpha}_m) + (1+\alpha)\frac{(1-\tau)(p-s)}{(1+\alpha)c-s}\bar{F}(q^{BWR}(x)) + \tau\bar{F}(d_T).\end{aligned}$$

The second line follows from (15). We drop the superscript BWR for notational convenience. The second derivative is

$$-\left((1-\tau)(1+\alpha)f(q)+\tau\frac{(1+\alpha)c-s}{p-s}f(d_T)\right)\frac{dq}{dx}+\tau\frac{\alpha}{p-s}f(d_T) \tag{16}$$

Using (13), (16) can be written as

$$\begin{aligned}\frac{d^2\Pi_2^S(x)}{dx^2} =&(1-\tau)\frac{\tau\alpha\left\{(p-s)-(1+\alpha)[(1+\alpha)c-s]\right\}f(d_T)f(q)+(1+\alpha)^2[(1+\alpha)c-s]f(q)f(d_B)}{(1-\tau)(p-s)^2f(q)-[(1+\alpha)c-s]^2(f(d_B)-\tau f(d_T))}\\ &+\frac{\tau[(1+\alpha)c-s]^2f(d_T)f(d_B)}{(p-s)\left[(1-\tau)(p-s)^2f(q)-[(1+\alpha)c-s]^2(f(d_B)-\tau f(d_T))\right]},\end{aligned}$$

which is non-negative because we assume that $\frac{p-s}{(1+\alpha)c-s}\geq 1+\alpha$ and the denominators of both terms are positive from (13). Hence, we can rule out the interior, (x_1,x_2), as the optimal solution will be at $x=x_1$ or $x=x_2$. Furthermore, the objective function is continuous, which implies that $x=x_1$ is taken in the account in the first case.

Proposition 4 *Let* $\alpha_l=\alpha\frac{(1-\tau)(p-s)}{(1+\alpha)c-s}\bar{F}(\beta x_1+\theta)+\bar{F}\left(\frac{(1+\alpha)c-s}{p-s}\theta\right)-1$ *and* $\alpha_h=(1+\alpha)\frac{(1-\tau)(p-s)}{(1+\alpha)c-s}\bar{F}(\theta)+\tau\bar{F}\left(\frac{(1+\alpha)c-s}{p-s}\theta\right)-1$. *The optimal equity investment with bankruptcy risk,* x_R^*, *is given by*

$$x_R^*=\begin{cases}x_1 & \text{if } \bar{\alpha}_m<\alpha_l,\\ \tilde{x}_R & \text{if } \bar{\alpha}_m\in[\alpha_l,\alpha_h],\\ 0 & \text{if } \bar{\alpha}_m>\alpha_h,\end{cases}$$

where $\tilde{x}_R$ *solves*

$$(1-\tau)\frac{(1+\alpha)(p-s)}{(1+\alpha)c-s}\bar{F}(\beta\tilde{x}_R+\theta)+\tau\bar{F}\left(\frac{\tilde{x}_R+[(1+\alpha)c-s]\theta}{p-s}\right)-1=\bar{\alpha}_m. \tag{17}$$

Proof of Proposition 4. This is Case 1 in which the owner solves

$$\Pi_1^{S^*}=\max_{x\in[0,x_1]}\Pi_1^S(x)=\max_{x\in[0,x_1]}(1+\bar{\alpha}_m)(K-x)+(p-s)\left[\int_{\frac{(1+\alpha)c-s}{p-s}\theta}^{q^L(x)}\bar{F}(\xi)d\xi-\tau\int_{d_T(x)}^{q^L(x)}\bar{F}(\xi)d\xi\right]$$

where $q^L(x)=\beta x+\theta$ and $d_T(x)=\frac{x+[(1+\alpha)c-s]\theta}{p-s}$. The objective function is concave in $[0,x_1]$ because the second derivative is

$$-(1-\tau)\frac{(1+\alpha)(p-s)}{(1+\alpha)c-s}\beta f(\beta x+\theta)-\frac{\tau}{p-s}f\left(\frac{x_1+[(1+\alpha)c-s]\theta}{p-s}\right)\leq 0.$$

Setting the first derivative equal to zero gives (17). α_h is obtained by setting $x=0$ in (17). Similarly, α_l is obtained by setting $x=x_1$. That is,

$$\begin{aligned}\alpha_l&=(1-\tau)\frac{(1+\alpha)(p-s)}{(1+\alpha)c-s}\bar{F}(\beta x_1+\theta)+\tau\bar{F}\left(\frac{x_1+[(1+\alpha)c-s]\theta}{p-s}\right)-1\\ &=(1-\tau)\frac{\alpha(p-s)}{(1+\alpha)c-s}\bar{F}(\beta x_1+\theta)+\bar{F}\left(\frac{(1+\alpha)c-s}{p-s}\theta\right)-1.\end{aligned}$$

The second equality follows from (14). It can be shown that $\tilde{x}_R<0$ when $\bar{\alpha}_m>\alpha_h$ and $\tilde{x}_R>x_1$ when $\bar{\alpha}_m<\alpha_l$. Moreover, $\alpha_h>\alpha_l$ because the left hand side of (17) is decreasing in x. This proves the result.

Proposition 5 *Let* $\alpha_2 = \alpha\left[1-\tau\bar{F}\left(\frac{x_2}{p-s}\right)\right]$ *and* $\alpha_3 = \alpha\left[1-\tau\bar{F}\left(\frac{c-s}{p-s}q^{BWO}(x_3)\right)\right]$. *The optimal equity investment in* $[x_2, K]$, *i.e., without bankruptcy risk, is*

$$x^*_{NR} = \begin{cases} x_2 & \text{if } \bar{\alpha}_m > \alpha_2, \\ x^*_3 & \text{if } \bar{\alpha}_m \in (\alpha_3, \alpha_2], \\ x^*_4 & \text{if } \bar{\alpha}_m \in [0, \alpha_3]. \end{cases}$$

where x^*_3 *solves* $\bar{F}\left(q^{BWO}(x^*_3)\right) = \frac{\bar{\alpha}_m[(1+\alpha)c-s]}{\alpha(1-\tau)(p-s)}$, *and* x^*_4 *solves* $(1-\tau)(p-s)\bar{F}\left(\frac{x^*_4}{c}\right)+\tau(c-s)\bar{F}\left(\frac{c-s}{p-s}\left[\frac{x^*_4}{c}\right]\right) = (1+\bar{\alpha}_m)c - s$.

Proof of Proposition 5. In Case $j \in \{3,4,5\}$, let $\Pi^S_j(x)$ denote the owner's payoff function as a function of the equity amount, and let $\Pi^{S^*}_j$ denote the owner's optimal payoff. In Case 3, the owner solves

$$\Pi^{S^*}_3 = \max_{x\in[x_2,x_3]} \Pi^S_3(x) = \max_{x\in[x_2,x_3]} (1+\bar{\alpha}_m)(K-x) + (1+\alpha)x + ((1+\alpha)c+s)q^{BWO}(x) + (p-s)\left[\int_0^{q^{BWO}(x)} \bar{F}(\xi)d\xi - \tau\int_{d_T}^{q^{BWO}(x)} \bar{F}(\xi)d\xi\right].$$

Taking a derivative with respect to x and using the definition of $q^{BWO}(x)$ from (7), we get

$$\frac{d\Pi^S_3}{dx} = \alpha - \bar{\alpha}_m - \alpha\tau\bar{F}\left(d_T(x)\right) = \alpha\frac{(1-\tau)(p-s)}{(1+\alpha)c-s}\bar{F}\left(q^{BWO}(x)\right) - \bar{\alpha}_m. \tag{18}$$

The second derivative with respect to x is $-\alpha\frac{(1-\tau)(p-s)}{(1+\alpha)c-s}\frac{dq^{BWO}(x)}{dx}f\left(q^{BWO}(x)\right)$, which is negative because q^{BWO} is increasing in x by Proposition 3. Therefore, the owner's problem is concave in Case 3. Concavity implies that (18) attains its maximum at $x = x_2$ and its minimum at $x = x_3$, which implies that the optimal solution is in (x_2, x_3) only if 0 is between the values of the first derivative at $x = x_2$ and $x = x_3$. The first derivatives at $x = x_2$ and $x = x_3$ are $\alpha\frac{(1-\tau)(p-s)}{(1+\alpha)c-s}\bar{F}\left(q^{BWO}(x_2)\right) - \bar{\alpha}_m$ and $\alpha\frac{(1-\tau)(p-s)}{(1+\alpha)c-s}\bar{F}\left(q^{BWO}(x_3)\right) - \bar{\alpha}_m$, respectively. Let $\alpha_2 = \alpha\frac{(1-\tau)(p-s)}{(1+\alpha)c-s}\bar{F}\left(q^{BWO}(x_2)\right) = \alpha\left[1-\tau\bar{F}\left(d_T\left(q^{BWO}(x_2), x_2\right)\right)\right]$. The last equality follows from (7). Similarly, let $\alpha_3 = \alpha\frac{(1-\tau)(p-s)}{(1+\alpha)c-s}\bar{F}\left(q^{BWO}(x_3)\right) = \alpha\left[1-\tau\bar{F}\left(d_T\left(q^{BWO}(x_3), x_3\right)\right)\right]$. Then the optimal equity investment in Case 3, x^*_3, is given by

$$x^*_3 = \begin{cases} x_2 & \text{if } \bar{\alpha}_m > \alpha_2, \\ \tilde{x}_3 & \text{if } \bar{\alpha}_m \in [\alpha_3, \alpha_2], \\ x_3 & \text{if } \bar{\alpha}_m < \alpha_3, \end{cases}$$

where $\tilde{x}_3$ is obtained by setting (18) equal to zero.

In Case 4, the owner solves

$$\Pi^{S^*}_4 = \max_{x\in[x_3,x_4]} \Pi^S_4(x) = \max_{x\in[x_3,x_4]} (1+\bar{\alpha}_m)(K-x) + \frac{s}{c}x + (p-s)\left[\int_0^{x/c} \bar{F}(\xi)d\xi - \tau\int_{\frac{(c-s)x}{(p-s)c}}^{x/c} \bar{F}(\xi)d\xi\right].$$

This objective function is concave in x because the second derivative is $-(1-\tau)\frac{p-s}{c^2}f(x/c) - \tau\left(\frac{c-s}{c}\right)^2\frac{1}{p-s}f\left(\frac{(c-s)x}{(p-s)c}\right) \leq 0$. The first derivative is

$$-(1+\bar{\alpha}_m) + \frac{s}{c} + (1-\tau)\frac{p-s}{c}\bar{F}\left(\frac{x}{c}\right) + \tau\frac{c-s}{c}\bar{F}\left(\frac{(c-s)x}{(p-s)c}\right), \tag{19}$$

which is decreasing in x. $\tilde{q}(x)$ is continuous by Proposition 3. Therefore, $x_3/c = q^{BWO}(x_3)$, and the first derivative at $x = x_3$ is

$$-(1+\bar{\alpha}_m) + \frac{s}{c} + \frac{p-s}{c}\left((1-\tau)\bar{F}\left(q^{BWO}(x_3)\right) + \tau\frac{c-s}{p-s}\bar{F}\left(\frac{c-s}{p-s}q^{BWO}(x_3)\right)\right). \tag{20}$$

Using (7), we know that $\tau\bar{F}\left(\frac{c-s}{p-s}q^{BWO}(x_3)\right) = 1 - \frac{(1-\tau)(p-s)}{(1+\alpha)c-s}\bar{F}(q^{BWO}(x_3))$. Substituting it into (20) gives $\alpha\frac{(1-\tau)(p-s)}{(1+\alpha)c-s}\bar{F}(q^{BWO}(x_3)) - \bar{\alpha}_m$, which is equal to $\alpha_3 - \bar{\alpha}_m$. Similar analysis shows that the derivative at $x = x_4$ is equal to $-\bar{\alpha}_m$. Therefore, $x_4^* = x_3$ if $\bar{\alpha}_m > \alpha_3$ and $x_4^* = \tilde{x}_4$ if $\bar{\alpha}_m \in [0, \alpha_3]$. where $\tilde{x}_4$ is obtained by setting (19) equal to zero.

In Case 5, the owner solves

$$\begin{aligned}\Pi_5^{S^*} &= \max_{x\in[x_4,K]} \Pi_5^S(x) \\ &= \max_{x\in[x_4,K]} (1+\bar{\alpha}_m)(K-x) + x - (c-s)\,q^{NB} + (p-s)\left[\int_0^{q^{NB}} \bar{F}(\xi)d\xi - \tau\int_{\frac{c-s}{p-s}q^{NB}}^{q^{NB}} \bar{F}(\xi)d\xi\right],\end{aligned}$$

which is linear in x. Therefore, $x_5^* = x_4$ because $\bar{\alpha}_m \geq 0$.

The objective function $\Pi(x)$ is concave in all three cases, and the first derivatives from the right and left are equal to each other at every switching point. Therefore, collecting these three cases together gives the optimal solution under no borrowing.

Proposition 6 *If $\Pi_R^{S^*}(0) > \Pi_{NR}^{S^*}(0)$ then investing with bankruptcy risk is optimal for all $\bar{\alpha}_m \geq 0$. Otherwise, there exists a unique threshold return value $\tilde{\alpha} \geq 0$ such that investing without bankruptcy risk is optimal when $\bar{\alpha}_m \leq \tilde{\alpha}$ and investing with bankruptcy risk is optimal otherwise.*

Proof of Proposition 6. We first show that both $\Pi_R^{S^*}$ and $\Pi_{NR}^{S^*}$ increase in $\bar{\alpha}_m$, but $\Pi_R^{S^*}$ increases at a faster rate. For $k \in \{R, NR\}$,

$$\frac{d\Pi_k^{S^*}}{d\bar{\alpha}_m} = \frac{\partial\Pi_k^S(\bar{\alpha}_m, x^*)}{\partial\bar{\alpha}_m} + \frac{\partial\Pi_k^S(\bar{\alpha}_m, x)}{\partial x}\Big|_{x=x^*}\frac{\partial x^*}{\partial\bar{\alpha}_m} = \frac{\partial\Pi_k^S(\bar{\alpha}_m, x^*)}{\partial\bar{\alpha}_m}.$$

This is due to $\frac{\partial\Pi_k^S(\bar{\alpha}_m,x)}{\partial x}\Big|_{x=x^*} = 0$ because x^* solves the first order condition of the owner's objective function. Therefore,

$$\frac{d\Pi_R^{S^*}}{d\bar{\alpha}_m} = \begin{cases} K - x_1 & \text{if } \bar{\alpha}_m < \alpha_l, \\ K - \tilde{x}_R & \text{if } \bar{\alpha}_m \in [\alpha_l, \alpha_h], \\ K & \text{if } \bar{\alpha}_m > \alpha_h, \end{cases}$$

and

$$\frac{d\Pi_{NR}^{S^*}}{d\bar{\alpha}_m} = \begin{cases} K - x_4^* & \text{if } \bar{\alpha}_m \in [0, \alpha_3], \\ K - x_3^* & \text{if } \bar{\alpha}_m \in (\alpha_3, \alpha_2], \\ K - x_2 & \text{if } \bar{\alpha}_m > \alpha_2. \end{cases}$$

In addition, one can show that $x_4^* > x_3^* > x_2$, $x_1 > \tilde{x}_R$, and $x_2 > x_1$. Also note that $K - x_4^* > 0$ because we assumed that the owner has sufficient capital to procure the optimal order quantity for a pure equity newsvendor. Therefore, $\frac{d\Pi_R^{S^*}}{d\bar{\alpha}_m} > \frac{d\Pi_{NR}^{S^*}}{d\bar{\alpha}_m} \geq 0$, which implies that if $\Pi_R^{S^*}(0) > \Pi_{NR}^{S^*}(0)$ then investing with risk optimal for all $\bar{\alpha}_m \geq 0$ because the two functions never intersect.

For a large $\bar{\alpha}_m$ value, investing with bankruptcy risk option leads to no investment in the newsvendor (i.e., $x_1^* = 0$). However, the owner invests x_2 if he chooses to invest without bankruptcy risk. Therefore,

$$\begin{aligned} \Pi_{NR}^{S^*} - \Pi_R^{S*} &= (1+\bar{\alpha}_m)(K - x_2) + (1+\alpha)x_2 - ((1+\alpha)c - s)q^{BWO}(x_2) \\ &\quad + (p-s)\left[\int_0^{q^{BWO}(x_2)} \bar{F}(\xi)d\xi - \tau \int_{d_T(q^{BWO}(x_2),x_2)}^{q^{BWO}(x_2)} \bar{F}(\xi)d\xi\right] \\ &\quad - \left((1+\bar{\alpha}_m)K + (p-s)\int_{\frac{(1+\alpha)c-s}{p-s}\theta}^{\theta} \bar{F}(\xi)d\xi\right) \\ &= -(1+\bar{\alpha}_m)x_2 + C_2 \end{aligned}$$

where C_2 is a constant. Hence, $\Pi_R^{S^*} > \Pi_{NR}^{S^*}$ for a sufficiently large $\bar{\alpha}_m$ value. Combining this result with $\frac{d\Pi_R^{S^*}}{d\bar{\alpha}_m} > \frac{d\Pi_{NR}^{S^*}}{d\bar{\alpha}_m} \geq 0$ implies that if $\Pi_R^{S^*}(0) \leq \Pi_{NR}^{S^*}(0)$ then there must exist a unique $\tilde{\alpha} \geq 0$ such that $\Pi_R^{S^*}(\tilde{\alpha}) = \Pi_{NR}^{S^*}(\tilde{\alpha})$. Hence, investing without bankruptcy risk is optimal when $\bar{\alpha}_m \leq \tilde{\alpha}$, and investing with bankruptcy risk is optimal otherwise.

Proposition 7 *Sequential optimization over $q(x)$ and x leads to the same expected ending cash position for the owner as joint optimization over q and x. That is, $\Pi^* = \Pi^{S^*}$.*

Proof of Proposition 7. In a joint optimization problem, the owner chooses (x, q) such that $0 \leq q \leq \beta x + \theta$. The constraint set can be divided into five subsets to determine the potential local optima.

i. When $q = \beta x + \theta$, the firm borrows with risk and the credit limit is binding. This is a single variable optimization problem, which is concave in x, and its optimal solution is presented in Proposition 4.

ii. When $\beta x < q < \beta x + \theta$, the firm borrows with risk, but the credit limit is not binding. The optimal solution cannot lie in this set because the owner's objective function in this interval is convex in x by Lemma 2.

iii. When $x/c < q \leq \beta x$, the owner's objective function is

$$\Pi(x,q) = (1+\bar{\alpha}_m)K + (\alpha - \bar{\alpha}_m)x - [(1+\alpha)c - s]q + (p-s)\left(\int_0^q \bar{F}(\xi)d\xi - \tau\int_{d_T(q,x)}^q \bar{F}(\xi)d\xi\right),$$

where $d_T(q,x) = \frac{(1+\alpha)c-s}{p-s}q - \frac{\alpha}{p-s}x$. Therefore,

$$\frac{\partial^2 \Pi}{\partial x^2} = -\tau(p-s)\left(\frac{\partial d_T}{\partial x}\right)^2 f(d_T) < 0,$$
$$\frac{\partial^2 \Pi}{\partial q^2} = -(1-\tau)(p-s)f(q) - \tau(p-s)\left(\frac{\partial d_T}{\partial q}\right)^2 f(d_T) < 0,$$
$$\frac{\partial^2 \Pi}{\partial x \partial q} = -(p-s)\tau\left(\frac{\partial d_T}{\partial q}\right)\left(\frac{\partial d_T}{\partial x}\right) f(d_T),$$

which imply that

$$\left(\frac{\partial^2 \Pi}{\partial x^2}\right)\frac{\partial^2 \Pi}{\partial q^2} - \left(\frac{\partial^2 \Pi}{\partial x \partial q}\right)^2 = (1-\tau)\tau(p-s)^2\left(\frac{\partial d_T}{\partial x}\right)^2 f(d_T)f(q) > 0.$$

Hence, the Hessian of Π is negative definite, which implies that Π is jointly concave in q and x. Thus, sequential optimization over $q(x)$ and x leads to the same solution as joint optimization over (x,q).

iv. When $q = x/c$, the firm does not borrow, and has no excess cash. This is a single variable optimization problem, which is concave in x, and its solution is presented in Proposition 5.

v. When $0 \le q < x/c$, the firm does not borrow, and has excess cash. This cannot be optimal because the owner can increase her expected return by investing the excess cash (i.e., $x - cq$) in the external market.

Hence, the optimal solution is either $q = \beta x + \theta$ or an (x,q) pair such that $q \in [x/c, \beta x]$. The owner's objective function is concave in both cases, which implies that $\Pi^* = \Pi^{S^*}$.

Lemma 3 *If the firm borrows with risk, then it borrows* $w = \frac{s}{1+\alpha}q + \left(c - \frac{s}{1+\alpha}\right)\theta$ *and the probability distribution of the bank's loan write-off is independent of* q*. In other words, let* $\mathcal{W} \equiv (p-s+b)[d_B - \xi]^+$ *denote the bank's loan write-off in case the firm defaults on its loan, and let the maximum of* $\mathcal{W}$ *be* $\mathcal{W}_{max} \equiv (p-s+b)\bar{G}^{-1}\left[1 - \left(\frac{\alpha}{1+\alpha}\right)\frac{p-s}{p-s+b}\right]$*. Then*

$$Pr(\mathcal{W} \le \zeta) = \begin{cases} \bar{F}\left(\bar{G}^{-1}\left[1 - \left(\frac{\alpha}{1+\alpha}\right)\frac{p-s}{p-s+b}\right] - \frac{\zeta}{p-s+b}\right) & \text{if } \zeta \le \mathcal{W}_{max}, \\ 1 & \text{if } \zeta > \mathcal{W}_{max}, \end{cases} \tag{21}$$

which is independent of x*.*

Proof of Lemma 3. If the firm uses its entire credit line, then $d_B = \frac{(1+\alpha)c-s}{p-s}(\beta x + \theta) - \frac{1+\alpha}{p-s}x = \bar{G}^{-1}\left[1 - \left(\frac{\alpha}{1+\alpha}\right)\frac{p-s}{p-s+b}\right]$. The bank's maximum loan write-off occurs when the realized demand is zero. Setting $\xi = 0$ in $\mathcal{W} = (p-s+b)[d_B - \xi]^+$ gives $\mathcal{W}_{max}$. For $0 \le \zeta \le \mathcal{W}_{max}$,

$$\begin{aligned} Pr\{\mathcal{W} \le \zeta\} &= Pr\{(p-s+b)[d_B - \xi]^+ \le \zeta\} \\ &= Pr\left\{(p-s+b)\left(\bar{G}^{-1}\left[1 - \left(\frac{\alpha}{1+\alpha}\right)\frac{p-s}{p-s+b}\right] - \xi\right) \le \zeta\right\} \\ &= \bar{F}\left(\bar{G}^{-1}\left[1 - \left(\frac{\alpha}{1+\alpha}\right)\frac{p-s}{p-s+b}\right] - \frac{\zeta}{p-s+b}\right). \end{aligned}$$

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